Time-response shaping using Output to Input Saturation Transformation

E. Chambon\textsuperscript{a,*}, L. Burlion\textsuperscript{a} and P. Apkarian\textsuperscript{a}

\textsuperscript{a}Onera – The French Aerospace Lab, 2 Avenue Édouard Belin, FR-31055, Toulouse, France

(Received 00 Month 20XX; accepted 00 Month 20XX)

For linear systems, the control law design is often performed so that the resulting closed-loop meets specific frequency-domain requirements. However, in many cases, it may be observed that the obtained controller does not enforce time-domain requirements amongst which the objective of keeping an output variable in a given interval. In this article, a transformation is proposed to convert prescribed bounds on an output variable into time-varying saturations on the synthesized linear control law. This transformation uses some well-chosen time-varying coefficients so that the resulting time-varying saturations do not overlap in the presence of disturbances. Using an anti-windup approach, it is obtained that the origin of the resulting closed-loop is globally asymptotically stable and that the regulated variable satisfies the time-domain constraints in the presence of an unknown finite-energy bounded disturbance. An application to a linear ball and beam model is presented.

Keywords: constrained control; linear systems; unknown disturbance; interval constraint; time-domain constraint; anti-windup

1. Introduction

To stabilise a given system, many techniques exist to obtain a control law satisfying to specified constraints. As far as MIMO systems are concerned, $H\infty$ loop-shaping can for example be used to enforce frequency-domain requirements. However it is possible that, using such control law, time-domain requirements on a so-called regulated variable $\alpha = C\alpha y \in \mathbb{R}$ are not fulfilled. This is illustrated on Fig. 1 where $\alpha$ time-response violates expected bounds $[\alpha(t), \overline{\alpha}(t)]$. In practice, a good knowledge of the studied system is often sufficient to shape its time-response. However, designing controllers satisfying to such prescribed time-domain requirements remains tedious and relies on numerous trial-and-errors involving simulations. Consequently, for more complex systems, and from the theoretical point of view, dedicated methods are often required to enforce both stability and time-domain constraints.

Amongst existing strategies to enforce time-domain requirements like time-response or overshoot limitation, it is possible to mention the work presented in Gevers (2002) which introduced the notion of Iterative Feedback Tuning (IFT). The idea is to shape the closed-loop in response to specific input signals so as to satisfy time-domain constraints. In the PID-tuning case, a comparison with practitioners methods was performed in Mossberg, Gevers, and Lequin (2002) which gives a hint on how to achieve time-domain requirements using this method. Time-domain specifications are also treated through optimal control strategies as extensively presented in Goodwin, Seron, and de Doná (2005). These approaches include model predictive control (MPC) in which the optimisation problem can take constraints into account, see for example Chen and Allgöwer (1996) or Chen and Allgöwer (1999). Computationally effective methods close to MPC are reported in Ghaemi, Sun, and Kolmanovsky (2012). The notion of reference-governor to adjust the reference trajectory so that the constraints on the system are satisfied is also noticeable. It was presented in Gilbert and...
Kolmanovsky (2002) with an application to aerospace systems in Polóni, Kalabić, McDonough, and Kolmanovsky (2014). The combination of frequency-domain and time-domain constraints has been explored in Apkarian, Ravanbod-Hosseini, and Noll (2011) and references therein. This method makes use of non-smooth bundle optimization methods and is referred to as “constrained structured H∞-synthesis”. It combines simulation optimization with H∞-synthesis to enforce both frequency- and time-domain requirements. Despite interesting numerical performance, this simulation-based technique does not guarantee the time-domain constraints satisfaction with respect to any type of input signals but those considered in the simulation.

These strategies often include the control law design, especially when an optimising scheme is used. Alternative schemes including anti-windup systems were proposed for example in Turner and Postlethwaite (2002) and Rojas and Goodwin (2002). Compared to the aforementioned results, the anti-windup design is interesting because the nominal control law remains unchanged when acting far from the constraints. Also, an extensive literature on the subject is available, see Tarbouriech and Turner (2009) or Galeani, Tarbouriech, Turner, and Zaccarian (2009) for instance. However, there is not necessarily a guarantee on the fact that the time-domain constraints will actually be satisfied. In this article, the approach presented in Burlion (2012) and applied in Burlion and de Plinval (2013) and Burlion, Poussot-Vassal, Vuillemin, Leitner, and Kier (2014) is presented in-depth for state-feedback minimum-phase linear systems subject to disturbances. The output–to-input–saturation transformation (OIST) theory proposes to reformulate prescribed bounds on the regulated variable α into state-dependent saturations on the control input u. This approach is illustrated on Fig. 2 where an ad hoc saturating block is inserted before the system control input. As indicated in this figure, additional information may be required to express these saturations. Using this method along with some assumptions, it is possible to obtain guarantees on the fulfilment of the time-domain constraints when an unsatisfactory control law has already been designed.

This article is an extension of the works presented in Burlion (2012) and Chambon, Burlion, and Apkarian (2015a) in the LTI framework. It gives a comprehensive description of the OIST method for minimum-phase linear systems with unknown finite-energy bounded disturbances, leading to
the OIST-LTI\textsuperscript{1} method. Further, it develops new results to bring guarantees to the method: the OIST design parameters are chosen time-varying to accommodate for saturations overlap\textsuperscript{2} and a solution is provided to enforce stability of the resulting saturated closed-loop.

The article is organised as follows: notations and definitions along with some functions properties are presented in Sect. 2 before introducing the ball and beam example. This example is used as a case study to highlight some problems linked to the application of OIST. Formal statements of the two considered problems are presented in Sect. 3 along with some assumptions which were used to obtain the results presented here. Then, the output to input saturation transformation proposed to solve the first problem is presented in Sect. 4. Due to the conservatism introduced by the bounds on the disturbances, special attention is paid in the selection of the design parameters so that the resulting time-varying saturations do not overlap. The second problem deals with the asymptotic stability of the origin of the system in closed-loop with the obtained saturated control. It is solved in Sect. 5 using an anti-windup structure. Finally, the whole approach is applied to the linear ball and beam model in Sect. 6. Conclusions and perspectives are then presented in Sect. 7.

2. Definitions and a motivating case study

2.1 Definitions and notations

2.1.1 Acronyms

The acronyms listed in Tab. 1 are used throughout the article.

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIBS</td>
<td>Converging Input Bounded State</td>
</tr>
<tr>
<td>CICS</td>
<td>Converging Input Converging State</td>
</tr>
<tr>
<td>GAS</td>
<td>Globally Asymptotically Stable (system)</td>
</tr>
<tr>
<td>GES</td>
<td>Globally Exponentially Stable (system)</td>
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<tr>
<td>ISS</td>
<td>Input-to-state Stable (system)</td>
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<tr>
<td>LQR</td>
<td>Linear-Quadratic Regulator</td>
</tr>
<tr>
<td>LTI</td>
<td>Linear Time-Invariant</td>
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<tr>
<td>MIMO</td>
<td>Multiple Inputs Multiple Outputs (system)</td>
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<tr>
<td>OIST</td>
<td>Output to Input Saturation Transformation</td>
</tr>
<tr>
<td>OIST-LTI</td>
<td>Output to Input Saturation Transformation for LTI systems</td>
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2.1.2 Notations

The state-space representation of the MIMO LTI system \((G)\) considered throughout this paper is given by:

\[
(G) \left\{ \begin{array}{l}
\dot{x} = Ax + Bu + Bd \\
y = x + De 
\end{array} \right.
\]

and is denoted \(G = (A, B, I_n, D)\) where \(A \in \mathbb{R}^{n \times n}\), \(B = \begin{bmatrix} B_u & B_d & 0 \end{bmatrix} \in \mathbb{R}^{n \times m}\) and \(D = \begin{bmatrix} 0 & 0 & D_e \end{bmatrix} \in \mathbb{R}^{n \times m}\). The state vector is denoted \(x \in \mathbb{R}^n\) and the measurements vector is denoted \(y \in \mathbb{R}^n\). The input vector lying in \(\mathbb{R}^m\) is divided between the single control input denoted

\textsuperscript{1}Output to Input Saturation Transformation for LTI systems.

\textsuperscript{2}For more details on how this notion is used in the following, the reader should refer to Def. 1.
\[ u \in \mathbb{R} \text{ and the unknown inputs which are denoted } [d, e] \in \mathbb{R}^{m-1} \text{ (respectively: state and measurement disturbances). Let denote } (K) \text{ such that } K = (A_K, B_K, C_K, D_K) \text{ a dynamic controller designed to achieve some frequency-domain constraints. Its state-space representation is given by} \]

\[
(K) \begin{cases}
x_K &= A_K x_K + B_K u_K \\
y_K &= C_K x_K + D_K u_K
\end{cases}
\]

where \( x_K \in \mathbb{R}^{n_K}, u_K = y \) and \( u = y_K \) – at least before applying OIST or considering additional stabilizing structures like anti-windups. The transfer function from an input \( u \) to an output \( y \) is denoted \( T_{u \rightarrow y}(s) \) where \( s \) is the Laplace variable.

Inequalities involving matrices of identical dimensions are understood component-wise: let \( (A, B) \in \mathbb{R}^{n \times m} \), then, \( A \leq B \iff \forall (i, j) \text{ s.t. } 1 \leq i \leq n, 1 \leq j \leq m, A_{ij} \leq B_{ij} \).

For a given bounded vector \( x(t), \) if the bounds are known they are denoted \( \underline{x}(t) \) and \( \overline{x}(t) \), i.e. \( \underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \).

The saturation and deadzone functions applied to a bounded variable \( x \) are respectively denoted \( \text{sat}_x(x) \) and \( \text{Dz}_x(x) \). They are related to each other by

\[
\text{sat}_x(x) = x - \text{Dz}_x(x)
\]

The standard Euclidean norm of a given signal \( x(t) \) defined for \( t \geq 0 \) is denoted \( \|x\| \). The \( L_2 \)-norm of the same signal is denoted \( \|x\|_2 \) and is given by

\[
\|x\|_2 = \sqrt{\int_0^\infty |x(t)|^2 \, dt}
\]

\[ 2.1.3 \quad \text{Definitions} \]

Some definitions are now introduced. In the article, the term “overlap” is used to refer to two signals taking the same value at a given instant and possibly changing order:

**Definition 1:** Two unidimensional signals \( s(t) \) and \( \pi(t) \) are said to overlap if \( \exists t_1 > 0, \delta > 0 \) such that \( \forall t < t_1, s(t) \leq \pi(t) \) and \( \forall t \in [t_1, t_1 + \delta] \), \( s(t) > \pi(t) \).

The notion is illustrated on Fig. 3. Next, the Lambert function is used to define some constants. It is defined as follows:

\[
\text{sat}_x(x) = x - \text{Dz}_x(x)
\]
Figure 4. Representation of the absolute value function (in black) and its differentiable approximate \( f_{\text{abs}} \) (in blue) over the interval \([-3, 3]\).

**Definition 2:** Let \( \forall x \in \mathbb{R}, \ F(x) = xe^x \). The inverse function of \( F \) is the Lambert function denoted \( W_0(y) \) which fulfills \( \forall y, \ F(W_0(y)) = W_0(y)e^{W_0(y)} = y \).

This definition is used to define constants which in turn will help to define differentiable approximates of non-differentiable functions:

**Definition 3:** Let define the constants \( \xi := \frac{1}{2}W_0\left(\frac{1}{e}\right) + \frac{1}{2} \) and \( \Xi := \xi - \tanh(\xi)\xi > 0 \). Using these notations, let also define the following functions, \( \forall (x, y) \in \mathbb{R}^2 \):

\[
\begin{align*}
  f_{\text{abs}}(x) &:= \tanh(x)x + \Xi \\
  f_{\text{max}}(x, y) &:= \frac{1}{2}[x + y + f_{\text{abs}}(x - y)] \\
  g(x, y) &:= f_{\text{max}}(f_{\text{abs}}(x), f_{\text{abs}}(y))
\end{align*}
\]

which definition is extended to vectors \((x, y) \in \mathbb{R}^{n \times 2}\) in a component-wise manner. The absolute value and its differentiable over-approximating function \( f_{\text{abs}}(x) \) are represented on Fig. 4 over an interval of \( \mathbb{R} \). This definition triggers some remarks:

**Remark 1:** Note that \( \xi \) is the solution to the equation \((2x - 1)e^{2x} = 1\). Also note that \( f_{\text{abs}}, f_{\text{max}} \) and \( g \) are continuous differentiable over \( \mathbb{R} \) or \( \mathbb{R}^2 \). Moreover,

- \( \forall x \in \mathbb{R}, \ f_{\text{abs}}(x) \geq |x| \);
- \( \forall (x, y) \in \mathbb{R}^2, \ f_{\text{max}}(x, y) \geq \max(x, y) \).

**Proof.** These inequalities can be proved using basic real analysis.

The following definitions are directly related to the implementation of the OIST method.

**Definition 4:** Let \( k \in \mathbb{N} \). Considering a LTI system \((G)\), a regulated output variable \( \alpha(t) = C_\alpha y(t) \in \mathbb{R} \) where \( C_\alpha \in \mathbb{R}^{1 \times n} \) is said to be of relative degree \( k \) with respect to \( u \) if and only if

\[
\forall i \text{ s.t. } 0 \leq i < k - 1, \ C_\alpha A^i B_u = 0 \text{ and } C_\alpha A^{k-1} B_u \neq 0
\]
2.2 Motivating case study

In this section, an example is introduced where the control synthesis problem has been solved without considering any time-domain constraint on the selected regulated output variable $\alpha$. In this case, there is a violation of the expected time-domain performance of this regulated variable which motivates the use of a dedicated method such as OIST. However, two problems related to the application of this method as presented in Burlion (2012) arise. This paper proposes solutions to both problems.

Note that a more thorough control design study may be sufficient to enforce the time-domain requirement. However, this is not considered in this article for two reasons:

- the OIST method was proposed to enforce time-domain requirements when the controller is not able to do so hence the failing controller is kept for illustrative purposes;
- other criteria often enter in the control design. Enforcing the time-domain criterion may degrade nominal performance from other points of view.

2.2.1 Considered ball and beam model

This case study is dedicated to the position control of a ball on a beam. The physical system and the notations are represented in Fig. 5.

The beam is actuated using a lever arm. An unknown disturbance force $d$ is eventually applied to the ball acceleration. The disturbance signal used in simulation is represented in plain blue on Fig. 6. The state vector of the system is defined by $x = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}$ and the measurements vector by $y = x$. The regulated variable is defined as:

$$\alpha = C_\alpha y = C_\alpha x$$

with $C_\alpha = \begin{bmatrix} 1 & 0 \end{bmatrix}$. It corresponds to the ball position.

The reason for monitoring this variable is quite obvious. The beam length is limited to $L = 1$ m which means that even a theoretically stabilizing control law can result in the ball falling off the beam especially in the presence of a disturbance. The time-domain requirement is thus to satisfy $0.1 \leq \alpha(t) \leq 0.9$ (in meters), $\forall t$ while driving the system from $r_0 = 0.5$ m to the setpoint.
Figure 6. In blue, considered state disturbance $d$ (in simulation). Known bounds $\overline{d}$ and $\underline{d}$ on this signal are represented in black.

(a) Time-domain requirement (in black) and simulation results for the regulated variable $\alpha$. The dashed-dotted red line is obtained when using the nominal controller and the plain red line is obtained when using OIST as in Burlion (2012).

(b) Control signal (in red) and control saturations obtained using OIST as in Burlion (2012). Overlap starts around $t_1 = 73s$.

Figure 7. Case study: simulation results w/ (plain red) or w/o (dashed-dotted red) OIST in the loop.

$r_s = 0.6m$. More exotic time-varying requirements can also be considered as illustrated below. The system state-space representation is given by:

$$
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 
B = \begin{bmatrix} 0 & 0 \\ -0.21 & 1 \end{bmatrix}, 
C = I_2, 
D = \begin{bmatrix} 0 & 0 \\ -0.3156 & -2.0937 \end{bmatrix}
$$

(8)

where the inputs are respectively $u$ and $d$. As far as the nominal control design is concerned, a state-feedback controller with integral action is implemented to achieve steady-state accuracy. The considered controller state-space representation is given by Eq. (2) where

$$
A_K = -1.5779, 
B_K = \begin{bmatrix} 0.0322 & 0.2339 \end{bmatrix}, 
C_K = -0.0644, 
D_K = \begin{bmatrix} -0.3156 & -2.0937 \end{bmatrix}
$$

(9)

and $u_K = y_s - y$ with $y_s = [r_s \ 0]^T$.

This dynamic controller stabilizes the system and yields good results on $r_s$-setpoint tracking. However, using this controller, the regulated variable represented in dashed-dotted red line in Fig. 7(a) violates the time-varying time-domain requirement represented in black lines.
2.2.2 Application of the original OIST method

To enforce the time-domain requirement, it is proposed to use the OIST approach as presented in Burlion (2012). In this article, the author uses a specific transformation which allows to transform a constraint on an output into saturations on the input. By appropriately choosing the saturations, the reachable set is restricted such that the regulated variable satisfies the considered requirement even in the presence of unknown disturbances. The only required information are bounds \( \underline{d} \) and \( \overline{d} \) on the disturbance \( d \) as illustrated in Fig. 6. For the sake of simplicity, the formulas are not described in this section. Please refer to Sect. 4 for a complete description of the method.

In the mentioned article, the design parameters used in the method are chosen constant. Let consider \( \kappa_1 = 1 \) and \( \kappa_2 = 0.6 \). The OIST approach is then implemented on the system. Simulation results are represented in plain red line in Fig. 7. At first, it seems that the regulated variable satisfies the requirement. However, it appears that violation of the constraint occurs at \( t = 75s \). This results from the two control saturations overlap starting at time \( t_1 \) as illustrated in Fig. 7(b). This could be avoided using time-varying coefficients \( \kappa_1(t) \) and \( \kappa_2(t) \). This was mentioned in Burlion (2012) but not detailed. The first main contribution of the current work is thus to propose guarantees within the OIST method such that saturations overlap never occurs and the time-domain requirement is guaranteed (see Sect. 4).

Moreover, inserting saturations in the loop is never harmless, especially when the controller is unstable. In most cases however, theoretical guarantees on the closed-loop stability are expected. This is the second main contribution of the paper (see Sect. 5).

3. Problems statement

Let consider the LTI system \((G)\) in Eq. (1) in closed-loop with the controller \((K)\) in Eq. (2) \((u_k = y \text{ and } u = y_K)\). Both the state and measurements are supposed to be disturbed. A time-domain constraint is expressed on a regulated variable \( \alpha \in \mathbb{R} \) defined as:

\[
\alpha = C_\alpha y = C_\alpha x + C_\alpha D_\alpha e = C_\alpha x + D_\alpha e
\]  

(10)

It consists in ensuring \( \alpha(t) \in [\underline{\alpha}(t), \bar{\alpha}(t)] \), \( \forall t \) where \( \underline{\alpha}(t) \) and \( \bar{\alpha}(t) \) are design parameters such that \( \bar{\alpha}(t) \geq \alpha(t) \), \( \forall t \). Before stating the tackled problems, some assumptions are now introduced.

3.1 Assumptions

To be able to use the transformation presented in Sect. 4, the considered system has to fulfil some assumptions which are recalled here. These assumptions are capital to be able to provide solutions to the considered problems. For example, it would be fanciful to enforce the time-domain constraint without having more information on the disturbances. First, the design signals of the time-domain constraint need to converge:

**Assumption 1**: The time-domain requirement signals are supposed to converge towards constant values:

\[
\lim_{t \to \infty} \underline{\alpha}(t) = \underline{\alpha}^*, \lim_{t \to \infty} \bar{\alpha}(t) = \bar{\alpha}^*
\]

(11)

where \( \underline{\alpha}^* \leq \bar{\alpha}^* \).

Second, the relative degrees of the regulated variable with respect to the inputs are detailed:
Assumption 2: Let \((k, l) \in \mathbb{N}^2\) such that \(1 \leq l \leq k\). It is supposed the regulated variable \(\alpha\) is of relative degree \(k\) (resp. \(l\)) with respect to \(u\) (resp. the state disturbance input \(d\)).

Note that since \(D_\alpha\) is supposed to be non-null then the regulated variable \(\alpha\) is of null relative degree with respect to the measurement noise \(e\). Let \(D(t) = [d, \dot{d}, \ldots, d^{(k-l-1)}]^{T} \in \mathbb{R}^{k-l}\) and \(E(t) = [e, \dot{e}, \ldots, e^{(k)}]^{T} \in \mathbb{R}^{k+1}\). The next assumption makes sure that these quantities are bounded by known time-varying matrices.

Assumption 3: Continuous time-varying bounds \([D(t), \overline{D}(t)]\) on the unknown disturbances and their derivatives \(D(t)\) are supposed to be known, that is

\[
D(t) \leq D(t) \leq \overline{D}(t), \forall t
\]

or, more precisely,

\[
\forall i \text{ s.t. } 0 \leq i \leq k - l - 1, \quad d^{(i)}(t) \leq \overline{d}^{(i)}(t) \leq \overline{d}^{(i)}(t), \forall t
\]

The same holds for the measurement disturbance and its derivatives \(E(t)\) with time-varying bounds \([E(t), \overline{E}(t)]\). Also:

\[
\lim_{t \to \infty} D(t) = D^*, \quad \lim_{t \to \infty} \overline{D}(t) = \overline{D}^*, \quad \lim_{t \to \infty} E(t) = E^* \quad \text{and} \quad \lim_{t \to \infty} \overline{E}(t) = \overline{E}^*
\]

where \(D^* = [d^* \quad 0 \quad \ldots \quad 0]^{T}\), \(\overline{D}^* = [\overline{d}^* \quad 0 \quad \ldots \quad 0]^{T}\), etc.

As conservative as this assumption may be, it is not so different from supposing the disturbances follow some theoretical model. The following assumptions will be used in the proof of the closed-loop system state convergence to the origin:

Assumption 4: The disturbance \(d\) is supposed to be of finite energy \(\|d\|_2 < \infty\). The same holds for \(e\) with energy \(\|e\|_2 < \infty\).

Assumption 5: The controller in Eq. (2) is supposed to stabilize asymptotically (resp. exponentially) system \((G)\) to the origin \(x^* = 0\), under Assum. 4 (resp. \(d = 0\) and \(e = 0\)). The controller state at the equilibrium is denoted \(x_K^* = 0\).

In this article, only minimum-phase systems are considered. In the non-minimum-phase case, additional analysis is required to ensure stability, which is considered as a perspective for future works.

Assumption 6: The zeros of the SISO transfer function \(T_{u \to \alpha}(s)\) from the control input \(u\) to the regulated variable \(\alpha\) are supposed to be with strictly negative real part.

To simplify the formulation of iterative expressions for the saturations components, the following assumption is made to ensure some terms will not re-appear upon derivation of the components:

Assumption 7: The relative degrees in Assum. 2 satisfy to the relation \(2l > k\).

For any system fulfilling these hypotheses, the two following problems are considered:

- Enforce the considered time-domain requirement on the considered regulated variable. This
is formulated in Pb. 1 and a solution is provided in Sect. 4. As illustrated in Sect. 2.2, this implicitly requires to avoid saturations overlap;

- Guarantee the closed-loop stability in the presence of OIST saturations. This is formulated in Pb. 2 and a solution is provided in Sect. 5.

### 3.2 Considered problems

The problem of enforcing this time-domain constraint was introduced in Burlion (2012) and translated in the linear framework in Chambon et al. (2015a):

**Problem 1** (Guaranteed satisfaction of a time-domain requirement, LTI framework): Find \([u(t), \overline{u}(t)]\) and \(C_0\) such that for two design signals \(a(t)\) and \(\overline{a}(t)\) fulfilling

\[ a(t) \leq \overline{a}(t), \forall t \]

for the system \((G)\) in Eq. (1) in closed-loop with controller \((K)\) in Eq. (2):

\[
\begin{align*}
\dot{x}(t) &= A\alpha x(t) + B_a u(t) + B_d d(t) \\
y(t) &= x(t) + D_e e(t) \\
\alpha(t) &= C_\alpha x(t) + D_\alpha e(t) \\
\dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\
u(t) &= C_K x_K(t) + D_K y(t) \\
u(t) &\in [u(t), \overline{u}(t)] \\
x_0 &\in C_0
\end{align*}
\]

(16)

where \(\alpha \in \mathbb{R}\) is called the “regulated” variable.

Finding a solution to this problem motivates the Output to Input Saturation Transformation (OIST) where a constraint on the regulated output \(\alpha\) as in Eq. (15) is transformed into saturations \([u(t), \overline{u}(t)]\) on the control input \(u\).

**Remark 2:** The problem of avoiding saturations overlap may not be obvious when considering the chosen formulation for Pb. 1. However, as illustrated in Sect. 2.2, finding a solution to Pb. 1 implicitly requires to avoid such overlap. Otherwise, no guarantee can be offered.

A guaranteed solution of this problem is proposed in Sect. 4. The introduction of saturations is critical in most cases. In the presence of saturations, the state of the closed-loop system may diverge: this is the well-known windup effect. Hence, a second problem should be considered in addition to Pb. 1:

**Problem 2** (Guaranteed closed-loop stability using OIST): Guarantee that the origin of the saturated closed-loop in Eq. (16) is asymptotically stable.

A solution to this problem is proposed in Sect. 5 using the saturations provided by the OIST approach in Sect. 4.

### 4. OIST-LTI with saturations overlap avoidance

The Output to Input Saturation Transformation is presented in this section as a solution to Pb. 1 in the case of known LTI systems fulfilling Assum. 1 to 7. Time-varying design coefficients are
introduced in the obtained saturations for the first time so as to avoid overlap. Their expressions are derived from the known bounds on $D(t)$ and $E(t)$. This is detailed in the following sections.

### 4.1 Regulated variable differentiation and relative degree

Using Assum. 2 and 7, the $k$-th derivative of the regulated variable $\alpha$ in function of $u$ and $d$ has the following expression:

$$\alpha^{(k)}(t) = C_\alpha A^k x(t) + D_\alpha e^{(k)}(t) + C_\alpha A^{k-1} B_u u(t) + \sum_{j=l}^{k} C_\alpha A^{j-1} B_d d^{(k-j)}(t)$$  \hspace{1cm} (17)

By definition of the relative degree (Def. 4), the $k$-th derivative of the regulated variable $\alpha$ thus depends on the control input signal $u(t)$. In the next section, a lemma will be formulated so that properly differentiated design bounds on $\alpha$ can lead to saturations on $u$, using Eq. (17).

### 4.2 Fulfilling the time-domain requirement

Considering Pb. 1, the objective is to ensure $\alpha(t) \in [\underline{\alpha}(t), \overline{\alpha}(t)], \forall t$. In this section, it is shown how adequate constraints on the successive derivatives of $\alpha$ can be used to fulfil this requirement. Let consider a vector of known positive time-varying signals

$$\kappa(t) = [\kappa_1(t) \ldots \kappa_k(t)] \in \mathbb{R}_+^k$$  \hspace{1cm} (18)

which will act as “design parameters”. The following lemma is proposed and proved:

**Lemma 1:** Let define $\underline{\alpha}_0(t) := \underline{\alpha}(t)$, $\overline{\alpha}_0(t) := \overline{\alpha}(t)$ by convention and $\forall j$ s.t. $1 \leq j \leq k$, $\forall t$:

$$\underline{\alpha}_j(t) := k_j(t) \left( \underline{\alpha}_{j-1}(t) - \alpha^{(j-1)}(t) \right) + \underline{\alpha}_j-1(t)$$

$$\overline{\alpha}_j(t) := k_j(t) \left( \overline{\alpha}_{j-1}(t) - \alpha^{(j-1)}(t) \right) + \overline{\alpha}_j-1(t)$$

**Proof.** See Appendix A. \hfill $\square$

**Remark 3:** The time-varying signal $\kappa(t)$ components $\kappa_j(t)$ in Eq. (19) are the design parameters of the method. Their selection is crucial to avoid saturations overlap as detailed in Sect. 4.5.

**Remark 4:** Note this Lemma is still valid when introducing more conservative bounds $\underline{\beta}_j(t)$ and $\overline{\beta}_j(t)$ on $\alpha^{(j)}(t)$, i.e. satisfying for any given $j$ such that $1 \leq j \leq k$:

$$\underline{\alpha}_j(t) \leq \underline{\beta}_j(t), \overline{\beta}_j(t) \leq \overline{\alpha}_j(t), \forall t$$  \hspace{1cm} (20)
Also note these more conservative bounds are not necessarily defined by an iterative relation as in Eq. (19).

This lemma is directly inspired by Assumption 2 on the relative degree of $\alpha$ with respect to $u$. It can be used to enforce a time-domain constraint on $\alpha$ by considering a time-domain constraint on its $k$-th derivative $\alpha^{(k)}$. Since the latter depends on $u$ as highlighted in Eq. (17), appropriate saturations on the control input can be obtained. This is detailed in the next section.

### 4.3 Control saturations

Given Assum. 2, there is $C_{\alpha} A^{k-1} B_u \neq 0$. Let also suppose that $C_{\alpha} A^{k-1} B_u > 0$. Considering Lemma 1 and Eq. (17), and supposing that the expressions of $\alpha_k(t)$ and $\alpha_k(t)$ are known $\forall t$, the saturations to apply to the control input $u$ can be obtained. In the undisturbed case ($d(t) = 0$ and $e(t) = 0$, $\forall t$), these saturations would simply be given by:

$$u(t) = \frac{1}{C_{\alpha} A^{k-1} B_u} \left[ \alpha_k(t) - C_{\alpha} A^{k} x(t) \right]$$

$$\bar{u}(t) = \frac{1}{C_{\alpha} A^{k-1} B_u} \left[ \alpha_k(t) - C_{\alpha} A^{k} x(t) \right]$$

In the more general considered case, the saturations expressions should account for the presence of unknown disturbances. This is possible using Assumption 3:

$$u(t) = \frac{1}{C_{\alpha} A^{k-1} B_u} \left[ \alpha_k(t) - C_{\alpha} A^{k} x(t) + |D_{\alpha}| \max \left( \left| e^{(k)}(t) \right|, \left| e^{(k)}(t) \right| \right) \right]$$

$$\bar{u}(t) = \frac{1}{C_{\alpha} A^{k-1} B_u} \left[ \alpha_k(t) - C_{\alpha} A^{k} x(t) - |D_{\alpha}| \max \left( \left| e^{(k)}(t) \right|, \left| e^{(k)}(t) \right| \right) \right]$$

To be appropriately defined, the saturations in both Eqs. (21) and (22) should not overlap, i.e. $\bar{u}(t) \geq u(t)$, $\forall t$. This can be ensured using appropriate time-varying design parameters $\kappa(t)$. This is discussed in Sect. 4.5.

**Remark 5:** In case $C_{\alpha} A^{k-1} B_u < 0$ and to avoid loss of generality, proper re-ordering of $u(t)$ and $\bar{u}(t)$ is required. For a given input signal $u$, the saturating operator can be defined as follows:

$$\text{sat}(u)(t) := \max(\min(u(t), \bar{u}(t)), \min(u(t), max(u(t), \bar{u}(t))))$$

### 4.4 Determination of bounds on the regulated variable derivatives

In the previous section, expressions of the control saturations were obtained. Using these saturations, one can enforce the considered time-domain constraint as shown in Lemma 1. As seen in Eqs. (21) and (22), these saturations depend on two quantities $\alpha_k(t)$ and $\alpha_k(t)$ which are iteratively defined in Eq. (19). A finer study of these quantities is performed in this section.

For any integer $j$ such that $1 \leq j \leq k$, let define the two following vectors:
Let use the differentiable expressions of Eq. (25) with
\[ Saturations analysis \]
Eq. (B1) is discussed. This selection is critical to avoid saturations overlap.

They satisfy the following relations:
\[ \alpha \]
\[ Sect. 3.1, the following explicit expressions are obtained for \]
\[ However, the expressions obtained in Eq. (24) cannot be used since \]
\[ for application on the derivatives \]
\[ be bounded by known quantities. This is used along with Def. 3 to obtain differentiable known \]
\[ • U^j(t) = \begin{bmatrix} u_0^j(t) & \ldots & u_k^j(t) \end{bmatrix} \in \mathbb{R}^{1 \times (k+1)} \text{ where } u_i^j(t) = 1 \text{ and } \forall i > j, u_i^j(t) = 0; \]
\[ • V^j(t) = \begin{bmatrix} v_0^j(t) & \ldots & v_{k-l-1}^j(t) \end{bmatrix} \in \mathbb{R}^{1 \times (k-l)} \text{ where } \forall i > \max(-1, j - l - 1), v_i^j(t) = 0 \]
\[ v_{l-1}^{j+1}(t) = u_{l-1}(t)C_\alpha A^{j-l}B_d \]

Let \[ \mathcal{A} = \begin{bmatrix} \alpha & \alpha & \ldots & (\alpha)^{(k)} \end{bmatrix}^\top \in \mathbb{R}^{k+1}, \quad \mathcal{A} = \begin{bmatrix} \alpha & \alpha & \ldots & (\alpha)^{(k)} \end{bmatrix}^\top \quad \text{and} \quad \Theta = C_\alpha C_\alpha A \ldots C_\alpha A^k] \]
\[ \in \mathbb{R}^{(k+1) \times n}. Using the iterative definition in Eq. (19) and notations in \]
\[ Sect. 3.1, the following explicit expressions are obtained for \[ \alpha_j(t) \text{ and } \overline{\alpha}_j(t): \]
\[ \alpha_j(t) = U^j(t) \{ \mathcal{A}(t) - \Theta x(t) - D_\alpha e(t) \} + C_\alpha A_j x(t) + D_\alpha e^{(j)}(t) - V^j(t) D(t) \]
\[ \overline{\alpha}_j(t) = U^j(t) \{ \mathcal{A}(t) - \Theta x(t) - D_\alpha e(t) \} + C_\alpha A_j x(t) + D_\alpha e^{(j)}(t) - V^j(t) D(t) \]

The vectors \[ U^j \text{ and } V^j \] can be defined iteratively when applying Eq. (19) to Eq. (24). This is
detailed in Appendix B.

\[ \textbf{Remark 6:} \] Using the obtained iterative expressions and the fact that \[ u_0^j(t) = \kappa_1(t), \] one can
determine that \[ \forall j, 0 \leq j < k, u_j^{j+1}(t) = \sum_{\nu=1}^{j+1} \kappa_\nu(t). \]

However, the expressions obtained in Eq. (24) cannot be used since \[ D(t) \text{ and } e(t) \] are unknown
quantities. As stated in Remark 4, Lemma 1 is still valid if more conservative bounds are considered
for application on the derivatives \[ \alpha^{(j)}(t). \] Under Assumption 3, the disturbances are known to
be bounded by known quantities. This is used along with Def. 3 to obtain differentiable known
expressions instead of the original ones in Eq. (24):
\[ \beta_j(t) = U^j(t) \{ \mathcal{A}(t) - \Theta x(t) \} + |D_\alpha| g(e^{(j)}(t), e^{(j)}(t)) \left| D_\alpha^\top \right| + C_\alpha A_j x(t) \]
\[ \beta_j(t) = U^j(t) \{ \mathcal{A}(t) - \Theta x(t) \} - |D_\alpha| g(e^{(j)}(t), e^{(j)}(t)) \left| D_\alpha^\top \right| + C_\alpha A_j x(t) \]

As stated in Remark 4, these bounds are not longer defined by iterative expressions as in Eq. (19).
They satisfy the following relations:
\[ \alpha_j(t) \leq \beta_j(t), \quad \overline{\alpha}_j(t) \geq \overline{\beta}_j(t) \]
which is compatible with the use of Lemma 1, as already mentioned. In the next section, the
selection of the coefficients in \[ \kappa(t) \] which appear in the iterative definitions of \[ U^j \text{ and } V^j \]
in Eq. (B1) is discussed. This selection is critical to avoid saturations overlap.

4.5 Saturations analysis

Let use the differentiable expressions of Eq. (25) with \[ j = k \] in place of \[ \alpha_k(t) \text{ and } \overline{\alpha}_k(t) \] in Eq. (22).
This is then possible to ensure \[ \alpha^{(k)}(t) \in [\beta^k(t), \overline{\beta}^k(t)], \forall t \text{ so that, after using Lemma 1 and } \]
Remark 4, the time-domain requirement is fulfilled:

\[
\begin{align*}
\underline{u}(t) &= \frac{\beta_k(t) - C_\alpha A^k x(t) + |D_\alpha| \max \left( |e^{(k)}|, |\bar{e}^{(k)}| \right)}{C_\alpha A^{j-1} B_\alpha} + \sum_{j=0}^{k-1} |C_\alpha A^{j-1} B_\alpha| \max \left( |d^{(k-j)}|, |\bar{d}^{(k-j)}| \right) \\
\bar{u}(t) &= \frac{\beta_k(t) - C_\alpha A^k x(t) - |D_\alpha| \max \left( |e^{(k)}|, |\bar{e}^{(k)}| \right)}{C_\alpha A^{j-1} B_\alpha} - \sum_{j=0}^{k-1} |C_\alpha A^{j-1} B_\alpha| \max \left( |d^{(k-j)}|, |\bar{d}^{(k-j)}| \right)
\end{align*}
\]

The obtained saturations depend on the design parameters \(\kappa(t)\) from Eq. (18), \(A(t)\) and \(\bar{A}(t)\) and on the signals describing the limited knowledge of the disturbances: \(\bar{D}(t), \bar{D}(t), \bar{E}(t)\) and \(\bar{E}(t)\). Some further analysis is however required. As highlighted in the case study of Sect. 2.2, it is particularly important to detect possible saturations overlap – by choosing the \(\kappa(t)\)-signals wisely – or to study the reachability of the origin – a result which will be used to prove the closed-loop stability.

4.5.1 Saturations overlap

To study the possible saturations overlap, the successive differences \(\bar{\beta}_j(t) - \beta_j(t)\) are considered up to \(j = k - 1\). For \(j = k\), one also has to consider the additional terms in \(\bar{u}(t) - u(t)\), see Eq. (27).

Let define \(\Delta_0(t) := \bar{u}(t) - u(t)\) and, \(\forall 1 \leq j < k\), \(\Delta_j(t) := \bar{\beta}_j(t) - \beta_j(t)\). Thus,

\[
\Delta_j(t) = U^j(t) \left\{ A(t) - \bar{A}(t) \right\} - 2 f_{\alpha \text{abs}} \left( U^j(t) g \left( E(t), \bar{E}(t) \right) \right) D_{\alpha}^\top - 2 f_{\alpha \text{abs}} \left( V^j(t) g \left( \bar{D}(t), \bar{D}(t) \right) \right) - 2 |D_\alpha| g \left( e^{(j)}, \bar{e}^{(j)} \right)
\]

Considering the iterative definitions in Eq. (B1), the difference in Eq. (28) can be factorized by \(\kappa_j(t)\). The expression is straightforward to obtain and can be written in the following form:

\[
\Delta_j(t) = \kappa_j(t) \lambda^d_j(t) + \lambda^\alpha_j(t)
\]

where \(\lambda^d_j(t)\) and \(\lambda^\alpha_j(t)\) only depend on the coefficients \(\kappa_l(t)\) with \(1 \leq l < j\). Using Eq. (29), it seems possible to force \(\Delta_j(t) \geq 0\), \(\forall t\) by appropriately defining the coefficient \(\kappa_j(t)\). The following lemma recalls the conditions under which saturations overlap is avoided:

**Lemma 2:** Saturations overlap is avoided if

- \(\forall j \ s.t. \ 0 \leq j < k\), \(\Delta_j(t) \geq 0\), \(\forall t\);
- \(\Delta_k(t) \geq 2 |D_{\alpha}| \max \left( |e^{(k)}|, |\bar{e}^{(k)}| \right) + 2 \sum_{j=1}^{k-1} |C_\alpha A^{j-1} B_\alpha| \max \left( |d^{(k-j)}|, |\bar{d}^{(k-j)}| \right), \forall t\).

The last condition ensures that \(\bar{u}(t) - u(t) \geq 0\), \(\forall t\) after considering Eq. (27). A constructive definition of the coefficients \(\kappa_j(t)\) in Eq. (18) can be used to fulfil Lemma 2:

**Theorem 1:** Saturations overlap is avoided if

- \(\overline{u}(t) \geq \underline{u}(t)\) (which is assumed in Pb. 1) and \(\lambda^d_j(t) \neq 0, \forall t\);
- \(\forall 1 \leq j < k\), one ensures \(\Delta_j(t) \geq 0\) by choosing, \(\forall t\),
May 18, 2016 International Journal of Control

\[
\kappa_j(t) = \frac{\bar{\kappa}_j - \lambda^n_j(t)}{\lambda^d_j(t)}
\]

(30)

where \(\bar{\kappa}_j > 0\) is chosen such that \(\kappa_1(t) > \frac{1}{2}, \forall j > 1, \kappa_j(t) > 1\) and \(\lambda^d_{j+1}(t) \neq 0, \forall t\).

- for \(j = k\), one ensures

\[
\Delta_k(t) \geq 2|\mathbf{D}_\alpha| \max \left( |\mathbf{c}^{(k)}|, |\mathbf{e}^{(k)}| \right) + 2 \sum_{j=l}^{k} |\mathbf{C}_\alpha \mathbf{A}^{j-1} \mathbf{B}_u| \max \left( |\mathbf{d}^{(k-j)}|, |\mathbf{d}^{(k-j)}| \right)
\]

by choosing, \(\forall t\),

\[
\kappa_k(t) = \frac{1}{\lambda^d_k(t)} \left[ \bar{\kappa}_k - \lambda^n_k(t) + 2|\mathbf{D}_\alpha| \max \left( |\mathbf{c}^{(k)}|, |\mathbf{e}^{(k)}| \right) \right. \\
\left. + 2 \sum_{j=l}^{k} |\mathbf{C}_\alpha \mathbf{A}^{j-1} \mathbf{B}_u| \max \left( |\mathbf{d}^{(k-j)}|, |\mathbf{d}^{(k-j)}| \right) \right]
\]

(31)

where \(\bar{\kappa}_k > 0\) is chosen such that \(\kappa_k(t) > \frac{1}{2}, \forall t\).

**Proof.** Straightforward using Lemma 2 and Eq. (29). As far as the minimal values for the \(\kappa_j(t)\) are concerned, this is discussed in the proof of Prop. 2 in Appendix C.

**Remark 7:** Note that \(\kappa_k(t)\) is not differentiable with respect to \(t\) by definition but this is not required contrary to the other coefficients.

By the end of this section, a vector of design parameters \(\bar{\kappa} = [\bar{\kappa}_1 \ldots \bar{\kappa}_k]\) is obtained, ensuring no overlap and the differentiability of \(\kappa(t)\) (to the exception of \(\kappa_k(t)\)).

### 4.5.2 State-independent saturations

It is interesting to note that only one term depends on the state vector \(\mathbf{x}(t)\) in the expressions of the control input saturations obtained using the definitions of \(\gamma_k\) and \(\gamma_k\) in Eq. (25). Let define:

\[
\forall t, \mathbf{K}_{\text{OIST}}(t) := \begin{pmatrix} U^k(t) \Theta \\ \mathbf{C}_\alpha \mathbf{A}^{k-1} \mathbf{B}_u \end{pmatrix} \in \mathbb{R}^{1 \times n}
\]

(32)

It is observed that introducing saturations on the control input \(u\) is equivalent to saturating the signal \(v\) defined as:

\[
v(t) := u(t) + \mathbf{K}_{\text{OIST}}(t) \mathbf{y}(t) = u(t) + \mathbf{K}_{\text{OIST}}(t) \mathbf{x}(t) + \mathbf{K}_{\text{OIST}}(t) \mathbf{D}_e \mathbf{e}(t), \forall t
\]

(33)

by the following state-free saturations:

\[
\begin{align*}
\underline{v}(t) &= \underline{u}(t) + \mathbf{K}_{\text{OIST}}(t) \mathbf{x}(t) + |\mathbf{K}_{\text{OIST}}(t) \mathbf{D}_e| \max (|\mathbf{e}(t)|, |\mathbf{e}(t)|) \\
\overline{v}(t) &= \overline{u}(t) + \mathbf{K}_{\text{OIST}}(t) \mathbf{x}(t) - |\mathbf{K}_{\text{OIST}}(t) \mathbf{D}_e| \max (|\mathbf{e}(t)|, |\mathbf{e}(t)|)
\end{align*}
\]

(34)
Remark 8: Positivity of $v(t) - v(t)$, \( \forall t \) is ensured using similar considerations than in Th. 1.

Using \( v \) instead of \( u \) as the new input to system \((G)\) in Eq. (1), the saturated system becomes

$$
\dot{x}(t) = [A - Bu]K_{oist}(t)x(t) + Bu_{sat}\bar{v}(t) - BuK_{oist}(t)De(t) + Bd(t)
$$

(35)

4.5.3 Admissible asymptotic equilibrium

Using Eq. (27) and the definitions of the state-free saturations in Eq. (34), the saturations on the new input \( v \) can be obtained. Using Assum. 1, there is also:

$$
\lim_{t \to \infty} A(t) = \begin{bmatrix} a^* & 0 & \ldots & 0 \end{bmatrix}^T, \quad \lim_{t \to \infty} \bar{A}(t) = \begin{bmatrix} a^* & 0 & \ldots & 0 \end{bmatrix}
$$

(36)

Considering Th. 1 along with Assum. 4, it can be observed that the design signal \( \kappa(t) \) in Eq. (18) converges towards a constant value. Thus, this is also the case of vectors \( U_j(t) \) and \( V_j(t) \), \( \forall j \) s.t. \( 1 \leq j \leq k \), and \( \lim_{t \to \infty} K_{oist}(t) = K^*_{oist} \). Consequently, as far as the saturations in Eq. (34) are concerned and using Assum. 3, they tend towards finite values and the unsaturated state-free control becomes:

$$
v^* = C_Kx^*_K + (DK + K^*_{oist})(x^* + De^*) = 0
$$

(37)

In the following proposition it is shown that the origin is an admissible equilibrium under some condition. This condition can be evaluated during the analysis phase of the unconstrained closed-loop system.

Proposition 1: Let \( x^* \in \mathbb{R}^n \) and suppose Assum. 1, 3 and 4 are satisfied. In the non-restrictive case where \( x^* = 0 \), this is an admissible asymptotic equilibrium if

$$
v^* = 0 \in [\bar{v}^*, \bar{v}^*]
$$

(38)

or, more precisely, if:

$$
\frac{1}{C_{(A^k - 1)B_a}} \left[ u_0^{k^*} \bar{a}^* + |Da| \max(|e^*|, |\bar{e}^*|) + \left\{ |v_0^{k^*}| + |C_{(A^{k-1})B_d}| \max \left(|d^*|, |\bar{d}^*| \right) \right\} \right.
$$

$$
+ \left\{ |K_{oist}^*| \max(|e^*|, |\bar{e}^*|) \right\} \leq 0 \leq \left\{ |K_{oist}^*| \max(|e^*|, |\bar{e}^*|) \right\}
$$

$$
\frac{1}{C_{(A^k - 1)B_a}} \left[ u_0^{k^*} \bar{a}^* - |Da| \max(|e^*|, |\bar{e}^*|) + \left\{ |v_0^{k^*}| + |C_{(A^{k-1})B_d}| \max \left(|d^*|, |\bar{d}^*| \right) \right\} \right.
$$

(39)

Using this analysis on the saturations, the stability of the system in closed-loop with the saturated nominal controller is studied in the next section.

5. Guaranteed closed-loop stability using OIST-LTI

Due to the introduction of saturations, the controller state may diverge upon saturation of the control. This is the well-known “windup” effect. Hence, even if a solution to Pb. 1 was provided in

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3In the non-pathological case where \( d \) is a physical signal converging to zero under Assum. 4.
Sect. 4, there is actually no guarantee that the closed-loop system state won’t diverge, as required in Pb. 2.

In this section, a solution to Pb. 2 is provided. The stability of the system in closed-loop with the saturated control signal \( \tau(t) \) (\( C_K \dot{x}_K(t) + D_K y(t) \)) (where \( u(t) \) and \( \bar{u}(t) \) have been obtained using the approach presented in Sect. 4) is enforced using an anti-windup approach.

### 5.1 Closed-loop representation

As mentioned in Sect. 4.5.2, the control signal saturations depend on the system state \( x(t) \). Changing the control signal into \( v(t) = u(t) + K_o(t)y(t) \), the system studied in this section is equivalent to the system in Eq. (40) where the saturations on \( v(t) \) do not depend on the state vector anymore:

\[
\begin{align*}
\dot{x}(t) &= [A - B_u K_o(t)] x(t) + B_u \tau(t) - B_u K_o(t) D_e e(t) + B_d d(t) \\
y(t) &= x(t) + D_e e(t) \\
\dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\
v(t) &= C_K x_K(t) + (D_K + K_o(t)) y(t) \\
\alpha(t) &= C_\alpha x(t) + D_\alpha e(t)
\end{align*}
\]

The following lemma will be used to demonstrate the final theorem of this paper:

**Lemma 3:** \( \forall x(t) \in \mathbb{R}^n, \forall t, \) the function \(-K_o(t)x(t)\) is Lipschitz (with respect to \( x \)). Moreover, \(-K_o(t)x(t)\) is \( K_1 \)-Lipschitz, where \( K_1 = \max_t \| K_o(t) \| \in \mathbb{R} \).

**Proof.** Let define, \( \forall x \in \mathbb{R}^n, \forall t, f(x, t) = -K_o(t)x(t) \). Then, \( \forall x \in \mathbb{R}^n, \frac{\partial f}{\partial x}(x, t) = -K_o(t) \). Since \( K_o(t) \) is continuous \( \forall t \) by definition of the coefficients \( \kappa(t) \) and continuity of \( \bar{D}(t), \bar{D}(t), \bar{e}(t), \bar{e}(t) \) (see Assum. 3), the function \( f \) is continuously differentiable with respect to the state \( x \). This implies that \( K_o(t)x(t) \) is a Lipschitz continuous function with respect to \( x \). \( \square \)

### 5.2 Guaranteeing the closed-loop stability with an anti-windup approach

Due to the presence of a dynamic controller and saturations, unexpected closed-loop behaviour is expected. Anti-windup techniques have been widely studied and used to avoid behaviours like controller state divergence. Some of these techniques are presented in Grimm et al. (2003); Kapoor, Teel, and Daoutidis (1998); Tarbouriech and Turner (2009). The approach proposed in Menon, Herrmann, Turner, Bates, and Postlethwaite (2006) and Herrmann, Menon, Turner, Bates, and Postlethwaite (2010) deals with a specific class of nonlinear systems to which the system presented in Eq. (40) belongs. In this article, the anti-windup framework is used to enforce the closed-loop stability of the system in Eq. (40) where the time-varying gain \( K_o(t) \) and saturations are respectively given by Eq. (32) and Eq. (34). This provides an answer to Pb. 2.

**Proposition 2:** The open-loop system

\[
\dot{x}(t) = [A - B_u K_o(t)] x(t)
\]

is GES.
The proof is inspired by Herrmann et al. (2010).

Proof. See Appendix C.

To ensure asymptotic stability of the saturated closed-loop, it is necessary to use an anti-windup. Considering the system in Eq. (40), the following anti-windup with state $x_a \in \mathbb{R}^n$ is introduced

\[
\begin{align*}
\dot{x}_a(t) &= A x_a(t) + B u_a(t) \\
y_a &= x_a \\
u_a(t) &= -K_{oist}(t)y_a(t) - D \tilde{z}(t)(v(t)) \\
v_1(t) &= -[K(s) + K_{oist}(t)]y_a(t)
\end{align*}
\]

(42)

The control $v$ is then modified into

\[
v(t) = u(t) + K_{oist}(t)y(t) + v_1(t) \\
= C_K x_K(t) + D_K (y(t) - y_a(t)) + K_{oist}(t)(y(t) - y_a(t))
\]

(43)

The main result of this section is the following theorem which proves stability of the origin of the system in closed-loop with the saturated nominal controller. Both this theorem and its proof are inspired by Menon et al. (2006) and Herrmann et al. (2010).

**Theorem 2:** If Assum. 1 to 7 are satisfied\(^4\) (resulting in Th. 1 and Prop. 2), the origin of the closed-loop system consisting of the system in Eq. (35) – where the time-varying saturations are given in Eq. (34) – the control law in Eq. (43) and the anti-windup compensator given in Eq. (42) is GAS.

**Proof.** See Appendix D.

An illustration of the system in closed-loop with the anti-windup and saturating block is given in Fig. 8. This is the typical structure obtained when implementing OIST on a linear system.

6. Example: back to the case study

The ball and beam example which served as a case study in Sect. 2.2 is considered again for application of the method which was presented in Sect. 4.

**Remark 9:** In this example, there is no disturbance on the measurements. The theory presented in Sect. 5 would however be applicable for example in the case where $\alpha = C_\alpha x + e$ with $e$ an unknown but bounded disturbance.

6.1 Assumptions

In this section, the assumptions in Sect. 3.1 are reviewed in the case of the ball and beam example introduced in Sect. 2.2.

- As far as the relative degrees are concerned, $k = 2$ and $l = 2$ which fulfills Assum. 2 and 7;

\[^4\text{And excluding the pathological case where } d \text{ is a non-converging finite energy distribution, which is not a realistic physical case.}\]
• The disturbance and its bounds are represented on Fig. 6. They fulfil Assum. 3;
• Assum. 4 is satisfied;
• The state-feedback controller with integral action proposed in Eq. (9) asymptotically stabilises the ball and beam system. Hence, Assum. 5 is satisfied;
• The system is equivalent to a double-integrator with no transmission zero: $T_{u \rightarrow x}(s) = -\frac{0.21}{s^2}$. Assum. 6 is fulfilled.

Remark 10: Note that the set-point $r_s = 0.6$ m (and $\dot{r}_s = 0$ m/s) is not the origin of the system (it is still a feasible equilibrium). However, using some transformation equivalent to a translation, one can obtain a set-point on the origin of the system. Hence Th. 2 and its proof are still valid.

6.2 OIST-LTI implementation, with no saturations overlap

Using results in Sect. 4 and considering $\alpha = 0.1$ m, $\bar{\tau} = 0.9$ m, the following expressions are obtained for the successive $\Delta_i(t)$:

$$\left\{\begin{array}{l}
\Delta_0(t) = 0.8 \\
\Delta_1(t) = \kappa_1(t) \lambda_1^d(t) + \lambda_1^I(t) \\
\Delta_2(t) = \kappa_2(t) \lambda_2^d(t) + \lambda_2^I(t)
\end{array}\right. \quad (44)$$

where

$$\begin{align*}
\lambda_1^d(t) &= \bar{\tau}(t) - \alpha(t) \\
\lambda_1^I(t) &= \bar{\dot{\alpha}} - \dot{\alpha} \\
\lambda_2^d(t) &= \left[\begin{matrix} \kappa_1(t) & 1 \\ \kappa_1(t) & 0 \end{matrix}\right] \{\bar{\mathbf{A}}(t) - \mathbf{A}(t)\} \\
\lambda_2^I(t) &= \left[\begin{matrix} \kappa_1(t) & \kappa_1(t) & 1 \end{matrix}\right] \{\bar{\mathbf{A}}(t) - \mathbf{A}(t)\}
\end{align*} \quad (45)$$

Then, the values of the design signals $\kappa_i(t)$ are deduced from these expressions and Th. 1:
\[
\begin{align*}
\kappa_1(t) &= \frac{\bar{\kappa}_1 - \lambda_1(t)}{\lambda_1^2(t)} \\
\kappa_2(t) &= \frac{\bar{\kappa}_2 - \lambda_2(t) + 2|C_{\alpha}AB_u|\max(|d(t)|,|\bar{d}(t)|)}{\lambda_2^2(t)}
\end{align*}
\]

(46)

where \(\bar{\kappa} = [\bar{\kappa}_1 \quad \bar{\kappa}_2] = [0.5 \quad 5]\) are chosen so that the conditions in Th. 1 are satisfied. It is then possible to obtain saturations on the control signal. Note that \(C_{\alpha}AB_u = -0.21 < 0\) in this example so the operator in Eq. (23) is used to obtain the adequate saturations.

6.3 Guaranteed closed-loop stability

In this example, the controller is stable and using an anti-windup is not necessary. For illustrative purposes and to illustrate the action of such structure, an anti-windup is however designed following results in Sect. 5. The time-varying coefficient \(K_{\text{oist}}(t)\) is defined as follows:

\[
K_{\text{oist}}(t) = \frac{U^2(t)\Theta}{C_{\alpha}AB_u}
\]

(47)

where \(U^2(t) = [\kappa_2(t)\kappa_1(t) + \kappa_1(t)\kappa_1(t) + \kappa_2(t)]\) and \(\Theta = [C_{\alpha} \quad C_{\alpha}A \quad C_{\alpha}A^2]^T\). The simulation results w/ or w/o an anti-windup structure in the loop are compared in Sect. 6.4.2.

6.4 Simulations and results

Using the results in Sect. 6.2, simulations are performed over 100 s. The disturbance signal used in simulation is shown in Fig. 6.

6.4.1 Simulation results w/o anti-windup structure

The simulation results are represented on Figs. 9 and 10. The data are represented in dashed-dotted red when considering the nominal control law only (no saturations), in plain red when considering OIST with constant coefficients (see the case study in Sect. 2.2) and in plain blue when the saturations obtained using OIST are introduced in the closed-loop and the OIST coefficients are chosen time-varying as in Th. 1 or Eq. (46). As mentioned in Sect. 2.2, the synthesized controller is not efficient enough and the ball falls off the beam. Using OIST and the knowledge on the disturbances bounds, the time-domain constraint is satisfied and nominal performance is recovered whenever the constraint is not violated. Note that the proposed approach leads to some conservatism due to the lack of knowledge on \(d\), especially around \(t = 45s\). Also, some conservatism could be introduced by using differentiable upper-approximates of the absolute value and maximum functions.

Contrary to the constant coefficients case, it can be noted on Fig. 9(b) that the use of time-varying coefficients as defined in Th. 1 offers guarantees on the time-domain requirement satisfaction at all times. This results from the saturations not overlapping in this case, whatever the choice of \(\bar{\kappa}\).

It appears in Fig. 10(a) that the control law variations are much sharper in the saturated cases. This is a trade-off required for complying with the time-domain requirement. Optimization of the constants \(\bar{\kappa}\) may help to obtain less demanding although satisfying control laws. This is considered future works.
May 18, 2016 International Journal of Control ijc OIST

(a) System state using time-varying coefficients (in blue), using constant coefficients (plain red), or, without using OIST (dashed-dotted red).

(b) Regulated variable using time-varying coefficients (in blue), using constant coefficients (plain red), or, without using OIST (dashed-dotted red). A zoom is performed for $72 \leq t \leq 82s$. In the time-varying coefficients case, the requirement is guaranteed.

Figure 9. System state and regulated variable $\alpha = r$ simulation results.

(a) Control signal obtained using time-varying coefficients (in blue), using constant coefficients (plain red), or, without using OIST (dashed-dotted red).

(b) OIST-LTI design parameters $\kappa(t)$ obtained (time-varying case only) through application of Th. 1.

Figure 10. Control signal and OIST design parameters $\kappa(t)$ simulation results.

6.4.2 Comparison of simulation results w/ or w/o an anti-windup structure

In the previous section, satisfying results were obtained without using an anti-windup structure in the feedback loop. The influence of such structure is illustrated in Fig. 11. The simulation results obtained with (resp. without) the anti-windup in the feedback loop are represented in plain (resp. dashed-dotted) blue. It appears that the use of an anti-windup allows the control law to stay longer on the saturations. This results in the regulated variable sticking to the time-domain requirement limits. In an informal way, this means that the nominal performance is less degraded since the control law tries to copy the original control as much as possible.
6.5 Conclusions on the simulation results

The results obtained in simulation are highly satisfactory. Using time-varying design parameters in the OIST approach as proposed in Th. 1, a time-domain requirement on a given regulated variable can be fulfilled with guarantees. Moreover, the closed-loop stability is ensured as demonstrated in Sect. 5. In practice, using an anti-windup structure is not mandatory. However, it has been observed on the illustrating example that the regulated variable sticks to the requirement bounds to copy the original system response as much as possible when using such structure.

7. Conclusion

In this article, the problem of keeping a linear system output – or regulated variable – in an interval has been formalized. A solution based on a transformation from the output expected “saturation” to a saturation on the existing linear control input has been proposed. A constructive method to apply this transformation has been introduced. Time-varying saturations are obtained and used in closed-loop. Special attention has been paid to choose the time-varying design parameters $\kappa_i(t)$ in Eq. (18) in order to avoid saturations overlap. Also, using results from the anti-windup design community, the stability of the system in closed-loop with the resulting non-linear control has been guaranteed under the considered assumptions. An application to a linear ball and beam model has been proposed, showing satisfactory results and illustrating the influence of the anti-windup structure.

However, throughout this article, the specific class of minimum-phase linear systems has been considered. Also, it has been supposed that the whole state is measured. Future works will be dedicated to extend the approach to non-minimum phase systems and to systems with output feedback. Some hints on the last aspect were already drawn in Chambon, Burlion, and Apkarian (2015b). Optimization of the coefficients $\tilde{\kappa}$ will also be considered.

References


Appendix A. Proof of Lemma 1

The proof is performed iteratively with fixed $k$. Let $j$ such that $1 \leq j \leq k$. Suppose that $\alpha^{(k)}(t) \in [\alpha_k(t), \overline{\alpha}_k(t)] \Rightarrow \alpha^{(j)}(t) \in [\alpha_j(t), \overline{\alpha}_j(t)], \forall t$. Also, $\alpha^{(j-1)}(0) \in [\alpha_{j-1}(0), \overline{\alpha}_{j-1}(0)]$. Only the lower bound is considered. The demonstration is similar in the upper bound case. Suppose

$$\exists t_2 > 0, \alpha^{(j-1)}(t_2) < \alpha_{j-1}(t_2)$$  \hspace{1cm} (A1)

then, since $\alpha^{(j-1)}(0) \in [\alpha_{j-1}(0), \overline{\alpha}_{j-1}(0)]$ and, by continuity of $\alpha^{(j-1)}$ and $\alpha_{j-1}$,

$$\exists t_1, 0 < t_1 < t_2, \left\{ \begin{array}{l}
\alpha^{(j-1)}(t_1) = \alpha_{j-1}(t_1) \\
\forall t \in [t_1, t_2], \quad \alpha^{(j-1)}(t) \leq \alpha_{j-1}(t)
\end{array} \right.$$  \hspace{1cm} (A2)

But, using the recurrence hypothesis, the definition of $\alpha_j(t)$ and the fact that $\forall t, \kappa_j(t) \geq 0$, one obtains, $\forall t \in [t_1, t_2],$

$$\alpha^{(j)}(t) \geq \alpha_j(t) \geq \kappa_j(t) \left( \alpha_{j-1}(t) - \alpha^{(j-1)}(t) \right) + \alpha_{j-1}(t)$$  \hspace{1cm} (A3)

\[\alpha^{(j-1)}(t_2) - \alpha^{(j-1)}(t_1) \geq \alpha_{j-1}(t_2) - \alpha_{j-1}(t_1) \]  \hspace{1cm} (A4)

which contradicts Eq. (A1). In other words

$$\forall t > 0, \alpha^{(j-1)}(t) \geq \alpha_{j-1}(t)$$  \hspace{1cm} (A5)

which proves the lemma.

Appendix B. Vectors $U^j$ and $V^j$ iterative definitions

The vectors $U^j$ and $V^j$ are given by the following iterative expressions:
\[
U^0(t) = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}
\]

\[
\forall j \text{ s.t. } 1 \leq j \leq k, \quad U^j(t) = \kappa_j(t)U^{j-1}(t) + \dot{U}^{j-1}(t) + \sigma \left( U^{j-1}(t) \right)
\]

\[
\forall j \text{ s.t. } 0 \leq j \leq l, \quad V^j(t) = 0
\]

\[
\forall j \text{ s.t. } l < j \leq k, \quad V^j(t) = \kappa_j(t) \left( V^{j-1}(t) + C_\alpha A^{j-l-1} \left[ A^{l-1} B_d \ldots A^{2l-j} B_d \right] \right) + V^{j-1}(t) + \sigma \left( V^{j-1}(t) \right) + \left[ \sum_{w=0}^{j-l-1} u_{l+1+w}^{j-l-1} C_\alpha A^{l-1+w} B_d \ 0 \ldots 0 \right]
\]

where \( \sigma \left( U^{j-1}(t) \right) \) is the cyclic permutation of length \( k+1 \) on the elements of \( U^{j-1}(t) \):

**Definition 5:** Let \( S = [s_0 \ldots s_k] \in \mathbb{R}^{k+1} \). The function

\[
\sigma(S) = [\sigma(s_0) \ldots \sigma(s_k)]
\]

is called the cyclic permutation of length \( k+1 \) on the elements of \( S \).

**Proof.** Tedium rewriting of Eq. (24) in explicit form and using Eq. (19) to express \( \alpha_{j+1} \) for \( j \geq 0 \) starting with \( \alpha_0(t) = \alpha(t) \) leads to Eq. (B1). The same calculus is performed as far as the upper bound is concerned.

**Appendix C. Proof of Proposition 2**

Considering Eq. (33) and the system in Eq. (40), this problem is equivalent to studying the stability of the system \( \dot{x}(t) = Ax(t) + Bu(t) \) in closed-loop with \( v(t) = 0 \) or \( u(t) = -K_{\text{clos}}(t)x(t) \) (\( d(t) = 0 \) and \( e(t) = 0 \)). The transfer function between \( u \) and the regulated variable \( \alpha \) is given by

\[
\alpha = T_{u \rightarrow \alpha}(s) = \frac{s^{m_1} + p_1 s^{m_2} + \ldots + p_{m_1-1}s + p_{m_1}}{s^{m_2} + d_1 s^{m_3} + \ldots + d_{m_1-1}s + d_{m_1}}
\]

with \( k = n - m \) (see Assum. 2). Theoretically speaking, a minimum state-space representation of this transfer can be represented in the canonical form which can in turn be expressed as a chain of integrators in addition to the considered transfer zero dynamics, see (Hu, Lindquist, Mari, & Sand, 2012, Chapter 4). The chain of integrators is given by
\[
\begin{align*}
\dot{\alpha} &= \dot{\alpha} \\
\vdots \\
\dot{\alpha}^{(k-2)} &= \alpha^{(k-1)} \\
\dot{\alpha}^{(k-1)} &= \alpha^{(k)} \\
&= -U^k(t)\Theta x + C_\alpha A^k x
\end{align*}
\] (C2)

where the last equality is obtained by observing that \( u = -K_{\text{dist}}(t)x \) (and \( d = 0, e = 0 \)). Let \( \forall 0 \leq j \leq k - 1, \gamma_j = \alpha^{(j)} + U^j(t)\Theta x - C_\alpha A^j x \) and \( \Gamma = [\gamma_0 \ldots \gamma_{k-1}] \in \mathbb{R}^k \). Using Eq. (19) and (24) with null disturbances, the chain of integrators in Eq. (C2) can be re-written as

\[
\dot{\Gamma} = \begin{bmatrix} -\kappa_1 & 1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & -\kappa_{k-1} & 1 \\
0 & \ldots & 0 & -\kappa_k
\end{bmatrix} \Gamma
\] (C3)

This is completed by the zero dynamics as shown in (Hu et al., 2012, Chapter 4) which results in the open-loop transfer in Eq. (C1) being equivalent to the following state-space representation

\[
\begin{bmatrix} \dot{\Gamma} \\
Z
\end{bmatrix} = \begin{bmatrix} A_{\Gamma} & 0 \\
A_{Z\Gamma} & A_Z
\end{bmatrix} \begin{bmatrix} \Gamma \\
Z
\end{bmatrix}
\] (C4)

where

\[
A_Z = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 1 \\
-p_m & -p_{m-1} & \ldots & -p_2 & -p_1
\end{bmatrix}
\] (C5)

and \( A_{Z\Gamma} \) is the null matrix except for the coefficient \( A_{Z\Gamma} (n - k, 1) = 1 \). Considering Assum. 6, \( A_Z \) eigenvalues are with strictly negative real parts. As far as the dynamics of Eq. (C4) is concerned, the following candidate Lyapunov positive definite function is considered

\[
V(\Gamma, Z) = \frac{1}{2} \Gamma^\top \Gamma + \frac{\epsilon}{2} Z^\top Z
\] (C6)

where \( \epsilon \) is a positive constant. Then
\[
\dot{V}(\Gamma, Z) = \Gamma^\top A\Gamma + \epsilon Z^\top A Z + \epsilon Z^\top A Z \Gamma 
\]  
(C7)

where, using the logarithmic function concavity and the fact that \( \forall 1 \leq i \leq k, \gamma_{i-1} \gamma_i \leq |\gamma_{i-1} \gamma_i| \):

\[
\begin{align*}
\Gamma^\top A\Gamma & = - \sum_{i=0}^{k-1} \kappa_{i+1} \gamma_i^2 + \sum_{i=1}^{k-1} \gamma_{i-1} \gamma_i \\
& \leq - \sum_{i=0}^{k-1} \kappa_{i+1} \gamma_i^2 + \frac{1}{2} \sum_{i=1}^{k-1} \gamma_i^2 + \frac{1}{2} \sum_{i=0}^{k-2} \gamma_i^2 \\
\Gamma^\top A\Gamma & \leq - \Gamma^\top D_{VT}\Gamma \\
& \leq - \Gamma^\top \text{diag}(\kappa_1 - \frac{1}{2}, \kappa_2 - 1, \ldots, \kappa_{k-1} - 1, \kappa_k - \frac{1}{2}) \Gamma
\end{align*}
(C8)

so that \( D_{VT} \) is a positive definite diagonal matrix upon adapted selection of the positive time-varying coefficients \( \kappa_i(t) \). In the same vein

\[
\epsilon Z^\top A Z \Gamma \leq \frac{\epsilon \nu}{2} \left( A_{Z\Gamma}^\top Z \right)^\top A_{Z\Gamma}^\top Z + \frac{\epsilon}{2\nu} \Gamma^\top \Gamma 
\]  
(C9)

where \( \nu \) is a positive constant. It comes that

\[
\dot{V}(\Gamma, Z) \leq - \Gamma^\top D_{VT}\Gamma + \frac{\epsilon \nu}{2\nu} \Gamma^\top \Gamma + \epsilon \left( Z^\top A Z + \frac{\nu}{2} Z^\top \left( A_{Z\Gamma} A_{Z\Gamma}^\top \right) Z \right) 
\]  
(C10)

Using the notations \( \lambda_{\min}(M) \) and \( \lambda_{\max}(M) \) to denote the minimal and maximal real parts of the eigenvalues of the matrix \( M \), it is observed that

\[
Z^\top A Z + \frac{\nu}{2} Z^\top \left( A_{Z\Gamma} A_{Z\Gamma}^\top \right) Z \leq \lambda_{\max}(A_Z) Z^\top Z + \frac{\nu}{2} \lambda_{\max}(A_Z) \lambda_{\max}(A_Z) Z^\top Z 
\]  
(C11)

and

\[
- \Gamma^\top D_{VT}\Gamma + \frac{\epsilon \nu}{2\nu} \Gamma^\top \Gamma \leq - \lambda_{\min}(D_{VT}) \Gamma^\top \Gamma + \frac{\epsilon \nu}{2\nu} \Gamma^\top \Gamma 
\]  
(C12)

By choosing \( \nu = - \frac{\lambda_{\max}(A_Z)}{\lambda_{\max}(A_Z \Gamma A_{Z\Gamma}^\top)} > 0 \) (since \( A_{Z\Gamma} A_{Z\Gamma}^\top \) is positive semi-definite) and \( \epsilon = \nu \lambda_{\min}(D_{VT}) > 0 \) and by observing that the eigenvalues of \( A_Z \) are with strictly negative real parts (see Assum. 6) and \( D_{VT} \) is a positive definite matrix, one obtains

\[
\dot{V}(\Gamma, Z) \leq - \frac{1}{2} \lambda_{\min}(D_{VT}) \Gamma^\top \Gamma + \frac{1}{2} \lambda_{\max}(A_Z) Z^\top Z \\
\leq - \lambda_{\min}(D_{VT}) - \frac{1}{2} \lambda_{\max}(A_Z) V(\Gamma, Z) \\
\leq - k_1 V(\Gamma, Z) 
\]  
(C13)

where \( k_1 > 0 \). Also note that \( \dot{V}(0, 0) = 0 \). As a consequence, the candidate function \( V \) is a Lyapunov function and the open-loop system \( \dot{x} = [A - B_u K_{oist}(t)] x \) is GES.
Appendix D. Proof of Theorem 2

First, considering Prop. 1, the origin is a reachable equilibrium of the system in closed-loop with the saturated control signal. Thus, it is of some interest to study the asymptotic stability of this equilibrium. Using Eq. (3), the state equation in Eq. (35) can be re-written as

\[ \dot{x}(t) = Ax(t) + Bu \left[ -K_{oist(t)}x_a(t) + C_Kx_K(t) + D_K(y(t) - x_a(t)) - D_{\tilde{w}}(v(t)) + B_d d(t) \right] \]  

(D1)

Using a similar approach to Kapoor and Daoutidis (1999), let define \( e_x(t) := x(t) - x_a(t) \). It follows that

\[
\begin{align*}
\dot{e}_x(t) &= (A + BuD_K)e_x(t) + BuC_Kx_K(t) + B_d d(t) + BuD_KD_e e(t) \\
\dot{x}_K(t) &= A_Kx_K(t) + B_K e_x(t) + B_K D_e e(t)
\end{align*}
\]

(D2)

Let \( X = [e_x, x_K]^T \) and \( W = [d \ e]^T \), then

\[
\begin{align*}
\dot{X} &= \begin{bmatrix} A + BuD_K & BuC_K \\ B_K & A_K \end{bmatrix} X + \begin{bmatrix} B_d & BuD_KD_e \\ 0 & B_KD_e \end{bmatrix} \begin{bmatrix} d \\ e \end{bmatrix} \\
&= AX + BW
\end{align*}
\]

(D3)

Under Assum. 3 and 4, \( \|W\|_{2} \) is finite and \( \|W\| \) is bounded. It follows that \( \|X\|_{2} \) is finite and \( \|X\| \) converges to zero. In case \( W = 0 \), the state \( X \) converges exponentially to zero. Replacing \( v \) in Eq. (42) by its expression in Eq. (43), one obtains the following equation:

\[
\dot{x}_a(t) = [A - BuK_{oist(t)}]x_a(t)
\]

\[
- BuD_{\tilde{w}}(v(t))K_{oist(t)}e_x(t) + K_{oist(t)}D_e e(t) + D_K e_x(t) + D_KD_e e(t) + C_Kx_K(t)
\]

\[
\]  

(D4)

Considering Prop. 2, the open-loop system \( \dot{x}_a = [A - BuK_{oist(t)}]x_a(t) \) is exponentially stable. Thus, for some positive definite function \( V(x_a) \), there exists by the converse Lyapunov theorem, see Khalil (1996), constants \( \alpha_i, 1 \leq i \leq 4 \), such that

\[
\begin{align*}
\alpha_1 \|x_a\|^2 \leq V(x_a) &\leq \alpha_2 \|x_a\|^2 \\
\|\frac{\partial V(x_a)}{\partial x_a}\| &\leq \alpha_3 \|x_a\| \\
\frac{\partial V(x_a)}{\partial x_a} [Ax_a - BuK_{oist(t)}x_a] &\leq -\alpha_4 \|x_a\|^2
\end{align*}
\]

(D5)

Since, by Eq. (D4):

\[
\dot{V}(x_a) = \frac{\partial V(x_a)}{\partial x_a} [Ax_a - BuK_{oist(t)}x_a] \\
- \frac{\partial V(x_a)}{\partial x_a} BuD_{\tilde{w}}(v(t))K_{oist(t)}e_x(t) + D_K e_x(t) + C_Kx_K(t) + (D_K + K_{oist(t)})D_e e(t)
\]

(D6)
it comes

\[
\dot{V}(x_a) \leq -\alpha_4 \|x_a\|^2 + \alpha_3 \|x_a\| B_u \|x_a\| \left\| D_z e_x(t) (K_{\text{oist}}(t)e_x(t) + D_K e_x(t) + C_K x_K(t) + (D_K + K_{\text{oist}}(t)) D_e e(t)) \right\| \tag{D7}
\]

First, using Lemma 3 and \(e_x := x - x_a\), it is observed that since \(K_{\text{oist}}(t)x\) is \(K_1\)-Lipschitz then

\[
\|K_{\text{oist}}(t)e_x\| \leq K_1 \|e_x\|, \quad \forall t \tag{D8}
\]

Second, by property of the deadzone function, \(\left\| D_z e_x(t) (v(t)) \right\| \leq \|v(t)\|, \quad \forall t\). It comes that

\[
\dot{V}(x_a) \leq -\alpha_4 \|x_a\|^2 + \alpha_3 \|B_u\| \|x_a\| \left\{ [K_1 + \|D_K\|] \|e_x\| + \|D_e\| \|e\| + \|C_K\| \|x_K\| \right\} \leq -\alpha_4 \|x_a\|^2 + \alpha_3 \|B_u\| \|x_a\| \left[ K_1 + \|[D_K \ C_K]\] \right] \|X\| + \alpha_3 \|B_u\| \|x_a\| \left[ K_1 + \|D_K\| \right] \|D_e\| \|W\| \tag{D9}
\]

Using \(k_1 > 0\) and \(k_2 > 0\) such that

\[
k_1 \geq \alpha_3 \|B_u\| \left[ K_1 + \|[D_K \ C_K]\] \right] \in \mathbb{R} \tag{D10}
\]

\[
k_2 \geq \alpha_3 \|B_u\| \left[ K_1 + \|D_K\| \right] \|D_e\| \in \mathbb{R}
\]

then Eq. (D9) becomes

\[
\dot{V}(x_a) \leq -\alpha_4 \|x_a\|^2 + \|x_a\| \left[ k_1 \|X\| + k_2 \|W\| \right] \tag{D11}
\]

Applying the inequality \(2\varepsilon \|x_a\| \|X\| \leq \varepsilon^2 \|x_a\|^2 + \frac{\|X\|^2}{\varepsilon^2}\) for \(\varepsilon > 0\), Eq. (D11) is re-written as

\[
\dot{V}(x_a) \leq -\alpha_4 \|x_a\|^2 + \frac{1}{2} k_1^2 \varepsilon_1^2 \|x_a\|^2 + \frac{1}{2} k_2^2 \varepsilon_2^2 \|x_a\|^2 + \frac{k_1^2}{\varepsilon_1^2} \|X\|^2 + \frac{k_2^2}{\varepsilon_2^2} \|W\|^2 \leq \left( -\alpha_4 + \frac{1}{2} k_1^2 \varepsilon_1^2 + \frac{1}{2} k_2^2 \varepsilon_2^2 \right) \|x_a\|^2 + \frac{k_1^2}{\varepsilon_1^2} \|X\|^2 + \frac{k_2^2}{\varepsilon_2^2} \|W\|^2 \tag{D12}
\]

where \(\alpha_5 = \alpha_4 - \frac{1}{2} k_1^2 \varepsilon_1^2 - \frac{1}{2} k_2^2 \varepsilon_2^2 > 0\) if the constants \(\varepsilon_1\) and \(\varepsilon_2\) are chosen small enough so that \(\alpha_4 > \frac{1}{2} k_1^2 \varepsilon_1^2 + \frac{1}{2} k_2^2 \varepsilon_2^2\). Using (Isidori, 1999, Lemma 10.4.2, p.21), \(V\) is thus an ISS-Lyapunov function for the system

\[
\dot{x}_a = f_1 \begin{pmatrix} x_a, \ [X \ W] \end{pmatrix} \tag{D13}
\]

where \(f_1\) is a non-linear function adequately defined. According to (Isidori, 1999, Theorem 10.4.1, p.21), the system in Eq. (D13) is thus ISS. At the beginning of this proof, it has been shown that – for a specific class of bounded finite energy disturbances \(d\) and \(e - \|X\|_2\) is finite and \(\|X\|\) converges
to zero. Using a similar approach to the previous case, there exists $V_X$ and strictly positive constants $\beta_1$, $\beta_2$ such that $\dot{V}_X(X) \leq -\beta_1 \|X\|^2 + \beta_2 \|W\|^2$. This function is an ISS-Lyapunov function to the following system

$$\dot{X} = f_2(X, W) \quad (D14)$$

where $f_2$ is a linear function adequately defined. Using (Isidori, 1999, Theorem 10.5.2, p.34), it is possible to conclude that the cascade of systems in Eq. (D15) is ISS. The cascade is illustrated on Fig. D1. Note that in case $d = 0$ and $e = 0$ and using (Isidori, 1999, Corollary 10.5.3, p.35), the origin $(x_a, X) = (0, 0)$ is GAS for the cascade.

$$\begin{cases}
\dot{x}_a = f_1(x_a, [X \ W]) \\
\dot{X} = f_2(X, W)
\end{cases} \quad (D15)$$

Using the relation between ISS and CICS property, as stated in (Terrell, 2009, Theorem 16.4, p.373), it comes that the cascade in Eq. (D15) is CICS. Hence, using the theorem in Sontag (1989) (where CIBS property is a weaker property than CICS), the origin $(x_a, X) = (0, 0)$ is GAS for the cascade. This concludes the proof.