Output- to Input-Saturation Transformation: application to a class of linear systems

Emmanuel Chambon[1], Laurent Burlion[1] and Pierre Apkarian[1]

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Abstract: *This article presents ongoing work on the Output- to Input-Saturation Transformation (OIST). Given a linear system, a controller satisfying frequency requirements may not satisfy time-domain requirements on a considered regulated output. For example, one may expect an output variable to remain in a given interval. The proposed transformation is used to convert lateral constraints on the regulated output into saturations on the control. The OIST theory is here demonstrated and applied in the case of a specific class of linear systems. An application to the problem of crane control is presented.*

1 Introduction

To stabilize a given system, many techniques exist to obtain a control law satisfying specified constraints. For example, nonsmooth optimization techniques can be used to enforce frequency-domain requirements.

However it is possible that, using this control law, time-domain requirements on a so-called *regulated variable* $\alpha = C_\alpha x \in \mathbb{R}$ are not enforced. This is illustrated on Fig. 1 where α time-response violates expected bounds $[\alpha_{\min}(t), \alpha_{\max}(t)]$.

The Output- to Input-Saturation Transformation ($OIST$) theory was first detailed in [1, 2] and proposes to find a remedy to this by reformulating 'saturations' (or expected bounds) on the regulated variable α into saturations on the controller output u. Other strategies include [3, 4] which use anti-windup loops to constrain the state or outputs in the time-domain. $OIST$ is illustrated on Fig. 2 where a saturation block is inserted before the system control input. As illustrated on this figure, additional information may be required to express these saturations which will be demonstrated in the following.

In this paper, it is proposed to expose and demonstrate the theoretical background of the $OIST$ approach in the case of known linear systems satisfying to additional hypotheses. A stabilizing static state-feedback controller is considered to obtain the non-saturated control law. In case of a dynamic controller, hints to extend the solution to that problem include anti-windup loops for example presented in [5, 6, 7] to account for the designed input saturations.

The paper is organized as follows: the $OIST$ problem is exposed in Sect. 3 after some definitions in Sect. 2 The hypotheses which will be satisfied by all linear systems considered in this paper are detailed in Sect. 4 The demonstration of the transformation applied to this class of systems is done in Sect. 5 where the general expression of the saturations is also obtained. In Sect. 6, the approach is fully detailed on the linearised model of a simplified crane. Simulations are also performed and results commented. The article ends up in Sect. 7 with some comment on the ongoing work and perspectives.

[1]Département de Commande des Systèmes et Dynamique du Vol – Onera, Centre de Toulouse, 2 Avenue Édouard Belin, 31000 Toulouse, France, emmanuel.chambon@onera.fr

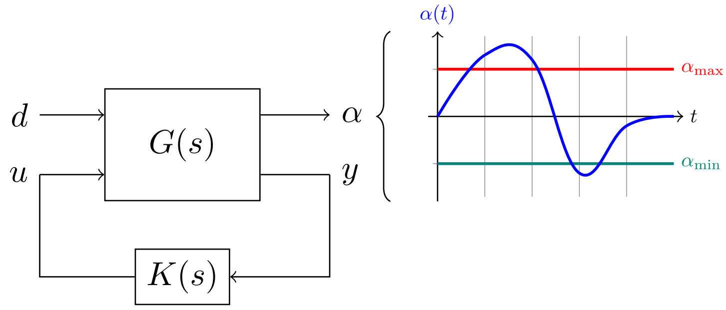

Figure 1: system (G) in closed-loop with controller K

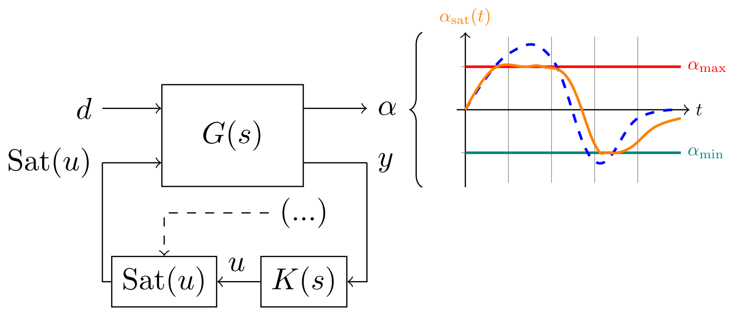

Figure 2: expected performance (in orange) with *OIST* block in the loop

2 Definitions and notations

Definition 1. *Let $k \in \mathbb{N}\backslash\{0\}$. Considering a system $G = (A, B, C, D)$ with state x and input u, the regulated variable $\alpha = C_\alpha x$ is said to be of relative degree k with respect to u if and only if*

$$\forall i \leq k-1,\ C_\alpha A^i B = 0 \text{ and } C_\alpha A^{k-1} B \neq 0 \tag{1}$$

In case $k = 0$, the dependence on u is direct: consider $\alpha = C_\alpha x + D_\alpha u$ with $D_\alpha \neq 0$. The following definition will be used to express iterative expressions on vector components in a simple manner

Definition 2. *Let $S = \begin{bmatrix} s_0 & \dots & s_k \end{bmatrix} \in \mathbb{R}^{k+1}$. The function $\sigma(S) = \begin{bmatrix} \sigma(s_0) & \dots & \sigma(s_k) \end{bmatrix}$ where*

$$\sigma(s_i) := \begin{cases} s_{i-i} & \text{if} \quad 1 \leq i \leq k+1 \\ s_{k+1} & \text{if} \quad i = 0 \end{cases} \tag{2}$$

is called the cyclic permutation of length $k+1$ on the elements of S.

3 Problem formulation

Let consider a known linear system (G) with the following state-space representation:

$$(G) \begin{cases} \dot{x} &=& Ax + B_u u + B_d d \\ y &=& Cx \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}$ is the control input and $d \in \mathbb{R}$ is an unknown disturbance. It is supposed a stabilizing control law $u = K(s)y$ has been designed for this system. The corresponding *OIST* problem is formalized into:

Problem 1. *Find $[u_{\min}(t), u_{\max}(t)]$ and $\mathcal{C}_0$ such that*

$$\alpha(t) \in [\alpha_{\min}, \alpha_{\max}],\ \forall t$$

for the system described as

$$\begin{cases} \dot{x} &= Ax + B_u u + B_d d \\ y &= Cx \\ \alpha &= C_\alpha x \\ u &\in [u_{\min}(t), u_{\max}(t)] \\ x_0 &\in \mathcal{C}_0 \end{cases} \tag{4}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}$ and $d \in \mathbb{R}$ is an unknown disturbance.

A solution to this problem in the case of known linear systems satisfying hypotheses in Sect. 4 is presented in Sect. 5.

4 Hypotheses

All the hypotheses satisfied by the systems considered in this article are presented here. Notations are those of Pb. 1.

Assumption 1. *Let $(k, l) \in \mathbb{N}^2$ such that $1 \leq l \leq k$. It is supposed the regulated variable α is of relative degree k (resp. l) wrt. u (resp. d).*

Assumption 2. *It is supposed that $k = l + 1$.*

Assumption 3. *Time-varying bounds $(\underline{d}, \overline{d}, \underline{\dot{d}}, \overline{\dot{d}})$ on the unknown disturbance d and its derivative $\dot{d}$ are supposed to be known, that is*

$$\forall t,\ \underline{d}(t) \leq d(t) \leq \overline{d}(t) \text{ and } \underline{\dot{d}}(t) \leq \dot{d}(t) \leq \overline{\dot{d}}(t)$$

Assumption 4. *The whole state is known that is $C = I_n$ and a static state-feedback $u = Kx$ will thus be used.*

Assumption 5. *The transfer $T_{u \to \alpha}(s)$ only has strictly negative zeros.*

Remark 1. *When saturated, a system with positive zeros may diverge away on the wrong side of the saturation. In this case, the approach is more complex and will be studied in the future.*

5 Proposed transformation

The output– to input–saturation transformation is presented in this section in the case of known linear systems satisfying Hyp. 1 to 5.

5.1 From the regulated variable to the input

Using Hyp. 1, it is possible to express the k-th derivative of the regulated variable α in function of u and d:

$$\alpha^{(k)} = C_\alpha A^k x + C_\alpha A^{k-1} B_u u + \sum_{j=l}^{k} C_\alpha A^{j-1} B_d d^{(k-j)} \tag{5}$$

which only depends on d and $\dot{d}$ under Hyp. 2. Using successive derivations, it is possible to transform bounds on α into bounds on u.

5.2 Keeping a variable in an interval

Let $\kappa(t) = \begin{bmatrix}\kappa_1(t) & \dots & \kappa_k(t)\end{bmatrix} \in \mathbb{R}_+^k$ a vector of positive time-varying signals, $\alpha_{0,\min} = \alpha_{\min}$ and $\alpha_{0,\max} = \alpha_{\max}$. Also consider $\mathcal{A}(t) = \begin{bmatrix}\alpha & \dots & \alpha^{(k)}\end{bmatrix}$ and $\Omega_{\min}(t) = \begin{bmatrix}\alpha_{0,\min} & \dots & \alpha_{k,\min}\end{bmatrix}$[1].

Lemma 1. *Let $\mathcal{A}(0) \in [\Omega_{\min}(0), \Omega_{\max}(0)]$ (element-wise) and $\forall j$, $1 \le j \le k$, $\forall t$:*

$$\begin{array}{rcl} \alpha_{j,\min}(t) & = & \kappa_j(t)\left(\alpha_{j-1,\min} - \alpha^{(j-1)}\right) + \dot{\alpha}_{j-1,\min} \\ \alpha_{j,\max}(t) & = & \kappa_j(t)\left(\alpha_{j-1,\max} - \alpha^{(j-1)}\right) + \dot{\alpha}_{j-1,\max} \end{array} \tag{6}$$

Then,

$$\alpha^{(k)} \in [\alpha_{k,\min}(t), \alpha_{k,\max}(t)]\,,\ \forall t \Rightarrow \alpha \in [\alpha_{\min}(t), \alpha_{\max}(t)]\,,\ \forall t \tag{7}$$

Proof. The proof is performed iteratively, with fixed k. Let $l \in \mathbb{N}$ s.t. $1 \le l+1 < k$. Suppose that $\alpha^{(k)}(t) \in [\alpha_{k,\min}(t), \alpha_{k,\max}(t)] \Rightarrow \alpha^{(l+1)}(t) \in [\alpha_{l+1,\min}(t), \alpha_{l+1,\max}(t)]$, $\forall t$. There is also $\forall t$, $\kappa_{l+1}(t) \ge 0$ and $\alpha^{(l)}(0) \in [\alpha_{l,\min}(0), \alpha_{l,\max}(0)]$.

Only the lower bound is considered. The demonstration is similar in the upper bound case. Suppose

$$\exists t_2 > 0,\ \alpha^{(l)}(t_2) < \alpha_{l,\min}(t_2) \tag{8}$$

then, since $\alpha^{(l)}(0) \in [\alpha_{l,\min}(0), \alpha_{l,\max}(0)]$ and by continuity of $\alpha^{(l)}$ and $\alpha_{l,\min}(t)$,

$$\exists t_1,\ 0 < t_1 < t_2,\ \left\{ \begin{array}{rrcl} & \alpha^{(l)}(t_1) & = & \alpha_{l,\min}(t_1) \\ \forall t \in [t_1, t_2]\,, & \alpha^{(l)}(t) & \le & \alpha_{l,\min}(t) \end{array} \right. \tag{9}$$

But, using the recurrence hypothesis, the definition of $\alpha_{l+1,\min}(t)$ and the fact that $\forall t$, $\kappa_{l+1}(t) \ge 0$, one obtains, $\forall t \in [t_1, t_2]$,

$$\begin{array}{rcl} \alpha^{(l+1)}(t) & \ge & \alpha_{l+1,\min}(t) \\ & \ge & \kappa_{l+1}(t)\left(\alpha_{l,\min}(t) - \alpha^{(l)}(t)\right) + \dot{\alpha}_{l,\min}(t) \\ & \ge & \dot{\alpha}_{l,\min}(t) \end{array} \tag{10}$$

hence, using the property of integrals

$$\begin{array}{rcl} \int_{t_1}^{t_2} \alpha^{(l+1)}(\lambda)\mathrm{d}\lambda & \ge & \int_{t_1}^{t_2} \dot{\alpha}_{l,\min}(\lambda)\mathrm{d}\lambda \\ \alpha^{(l)}(t_2) - \alpha^{(l)}(t_1) & \ge & \alpha_{l,\min}(t_2) - \alpha_{l,\min}(t_1) \end{array} \tag{11}$$

which contradicts Eq. (8). In other words

$$\forall t > 0,\ \alpha^{(l)}(t) \ge \alpha_{l,\min}(t) \tag{12}$$

□

5.3 Control saturations

From the previous lemma and using Hyp. 1[2], Hyp. 2 and the expression in Eq. (5), one can determine the saturations to apply to the controller output u before injection in the system. Note the expression is only developed for the lower bound but a similar approach is used in the upper bound case. The hypothesis $C_\alpha A^{k-1} B_u > 0$ was also made but switching the definitions of $u_{\min}$ and $u_{\max}$ solves the other case.

$$u_{\min}(t) = \frac{1}{C_\alpha A^l B_u}\left[\alpha_{l+1,\min}(t) - C_\alpha A^{l+1} x + \sum_{j=l}^{l+1} |C_\alpha A^{j-1} B_d| \max(|\underline{d^{(l+1-j)}}|, |\overline{d^{(l+1-j)}}|)\right] \tag{13}$$

[1] $\Omega_{\max}(t)$ is defined in a symmetrical manner.

[2] Especially: $C_\alpha A^{k-1} B_u \neq 0$ (see Def. 1).

Remark 2. *For this definition to be consistent and to avoid saturations crossings, one must ensure that*

$$\alpha_{l+1,\max}(t) - \alpha_{l+1,\min}(t) \geq 2\sum_{j=l}^{l+1} |C_\alpha A^{j-1} B_d| \max(|\underline{d^{(l+1-j)}}|, |\overline{d^{(l+1-j)}}|) \tag{14}$$

5.4 Determining $\alpha_{j,\min}$ and $\alpha_{j,\max}$ expressions

The expressions of the bounds on the successive derivatives of the regulated variable α can be determined using the iterative definition in Eq. (6). For the sake of space, only the lower bounds are considered. Determining the upper bounds is then straightforward.

For any integer j such that $1 \leq j \leq k$, let consider the two following vectors:

- $U^j(t) = \begin{bmatrix} u_0^j(t) & \dots & u_k^j(t) \end{bmatrix}$ where $u_j^j(t) = 1$ and $\forall i > j$, $u_i^j(t) = 0$;
- $V^j(t) = \begin{bmatrix} v_0^j(t) & \dots & v_{k-l-1}^j(t) \end{bmatrix}$ where: $\forall i > \max(-1, j-l-1)$, $v_i^j(t) = 0$ and $v_0^{l+1}(t) = u_l^{l+1}(t) C_\alpha A^{l-1} B_d$,

Let

$$\begin{aligned} \mathcal{A}_{\min} &= \begin{bmatrix} \alpha_{\min} & \dot{\alpha}_{\min} & \dots & \alpha_{\min}^{(k)} \end{bmatrix}^\top \\ \mathcal{D} &= \begin{bmatrix} d & \dot{d} & \dots & d^{(k-l-1)} \end{bmatrix}^\top \\ \Theta &= \begin{bmatrix} C_\alpha & C_\alpha A & \dots & C_\alpha A^k \end{bmatrix}^\top \end{aligned} \tag{15}$$

Using these definitions, one can express the lower bound $\alpha_{j,\min}(t)$ to be satisfied by $a^{(j)}$ as:

$$\alpha_{j,\min}(t) = U^j(t)\left\{\mathcal{A}_{\min} - \Theta x\right\} + C_\alpha A^j x - V^j(t)\mathcal{D}(t) \tag{16}$$

Then, using Eq. (6), the formulas in Eq. (17) are obtained where $\sigma\left(U^{j-1}(t)\right)$ is the cyclic permutation of length $k+1$ on the elements of $U^{j-1}(t)$.

$$\begin{array}{lrcl} & U^0(t) & = & \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \\ \forall j \text{ s.t. } 1 \leq j \leq k, & U^j(t) & = & \kappa_j(t) U^{j-1}(t) + \dot{U}^{j-1}(t) + \sigma\left(U^{j-1}(t)\right) \\ \forall j \text{ s.t. } 0 \leq j \leq l, & V^j(t) & = & 0 \\ \forall j \text{ s.t. } l < j \leq k-l, & V^j(t) & = & \kappa_j(t)\left(V^{j-1}(t) + C_\alpha A^{j-l-1}\begin{bmatrix} A^{l-1}B_d & \dots & A^{2l-k}B_d \end{bmatrix}\right) + \dot{V}^{j-1}(t) \\ & & & +\sigma\left(V^{j-1}(t)\right) + \begin{bmatrix} \sum_{w=0}^{j-l-1} u_{l-1+w}^{j-1}(t) C_\alpha A^{l-1+w} B_d & 0 & \dots & 0 \end{bmatrix} \end{array} \tag{17}$$

Proof. Writing Eq. (16) in explicit form and using Eq. (6) to express $\alpha_{j+1,\min}$ for $j \geq 0$ starting with $\alpha_{0,\min}(t) = \alpha_{\min}(t)$ leads to Eq. (17). □

Remark 3. *Using the previous iterative expressions and the fact that* $u_0^1(t) = \kappa_1(t)$, *one can easily determine that* $\forall j$, $0 \leq j \leq k$, $u_j^{j+1}(t) = \sum_{w=1}^{j+1} \kappa_w(t)$.

5.5 Avoiding crossings

Under Hyp. 2, $\forall j$, $0 \leq j \leq l$,

$$\alpha_{j,\min}(t) = U^j(t)\left\{\mathcal{A}_{\min} - \Theta x\right\} + C_\alpha A^j x$$

and, using the expression of $v_0^{l+1}(t)$ and remark 3,

$$\alpha_{l+1,\min}(t) = \left[\kappa_{l+1}(t)U^l(t) + \dot{U}^l(t) + \sigma\left(U^l(t)\right)\right]\{\mathcal{A}_{\min} - \Theta x\} + C_\alpha A^{l+1}x - \sum_{i=1}^{l+1}\kappa_i(t)C_\alpha A^{l-1}B_d d \quad (18)$$

Since d is an unknown disturbance and under Hyp. 3, this expression becomes

$$\begin{aligned}\alpha_{l+1,\min}(t) = {} & \left[\kappa_{l+1}(t)U^l(t) + \dot{U}^l(t) + \sigma\left(U^l(t)\right)\right]\{\mathcal{A}_{\min} - \Theta x\} + C_\alpha A^{l+1}x \\ & + \sum_{i=1}^{l+1}\kappa_i(t)\left|C_\alpha A^{l-1}B_d\right|\max\left(|\underline{d}|, |\overline{d}|\right) \quad (19)\end{aligned}$$

considering that $\forall i$, $\forall t$, $\kappa_i(t) > 0$. Let define $\Omega_j(t) = \alpha_{j,\max}(t) - \alpha_{j,\min}(t)$. Then, for $j = l+1 = k$:

$$\begin{aligned}\Omega_{l+1}(t) = {} & \kappa_{l+1}(t)\left[U^l(t)\{\mathcal{A}_{\max} - \mathcal{A}_{\min}\} - 2|C_\alpha A^{l-1}B_d|\max(|\underline{d}|, |\overline{d}|)\right] \\ & + \left[\dot{U}^l(t) + \sigma\left(U^l(t)\right)\right]\{\mathcal{A}_{\max} - \mathcal{A}_{\min}\} - 2\sum_{i=1}^{l}\kappa_i(t)|C_\alpha A^{l-1}B_d|\max(|\underline{d}|, |\overline{d}|) \quad (20)\end{aligned}$$

Due to the presence of eventually conservative bounds on the unknown disturbance d and depending on the parameter $\kappa_{l+1}(t)$, there could exist a time t_c such that

$$\Omega_k(t_c) = \Omega_{l+1}(t_c) < 2\sum_{j=l}^{k}|C_\alpha A^{j-1}B_d|\max(|\underline{d^{(k-j)}}|, |\overline{d^{(k-j)}}|) \quad (21)$$

which invalidates the whole theory through Lemma 1 and Remark 2. Thus, care must be taken in the definition of the coefficients of $\kappa(t)$. The following expressions are proposed:

- noting that $\forall t$, $\forall j$, $1 \leq j \leq l$, $\Omega_j(t) = \kappa_j(t)\Omega_{j-1}(t) + \dot{\Omega}_{j-1}(t)$, one can ensure $\Omega_j(t) > 0$ by choosing
$$\kappa_j(t) = \frac{\breve{\kappa}_j - \dot{\Omega}_{j-1}(t)}{\Omega_{j-1}(t)}$$
where $\breve{\kappa}_j > 0$ is a constant ensuring $\forall t$, $\kappa_j(t) \geq 0$. Note that $\alpha_{\min}(t)$ and $\alpha_{\max}(t)$ need to be wisely chosen to ensure $\Omega_0(t) > 0$.
- for $j = k = l+1$, one has to ensure $\forall t$, $\Omega_k(t) \geq 2\sum_{j=l}^{k}|C_\alpha A^{j-1}B_d|\max(|\underline{d^{(k-j)}}|, |\overline{d^{(k-j)}}|)$ (see remark 2). Using Eq. (20), the expression in Eq. (22) is obtained where $\breve{\kappa}_k > 0$ is a constant ensuring $\forall t$, $\kappa_k(t) \geq 0$.

$$\begin{aligned}\kappa_{l+1}(t) = {} & \frac{1}{U^l(t)\{\mathcal{A}_{\max} - \mathcal{A}_{\min}\} - 2|C_\alpha A^{l-1}B_d|\max(|\underline{d}|, |\overline{d}|)}\left\{\breve{\kappa}_k - \left[\dot{U}^l(t) + \sigma\left(U^l(t)\right)\right]\{\mathcal{A}_{\max} - \mathcal{A}_{\min}\}\right. \\ & \left. + 2\sum_{i=1}^{l}\kappa_i(t)|C_\alpha A^{l-1}B_d|\max(|\underline{d}|, |\overline{d}|) + 2\sum_{j=l}^{k}|C_\alpha A^{j-1}B_d|\max(|\underline{d^{(k-j)}}|, |\overline{d^{(k-j)}}|)\right\} \quad (22)\end{aligned}$$

6 Example

To illustrate the theoretical results presented in this article, the *OIST* approach is applied to the control of a crane of total mass $M + m = 0.6$kg.

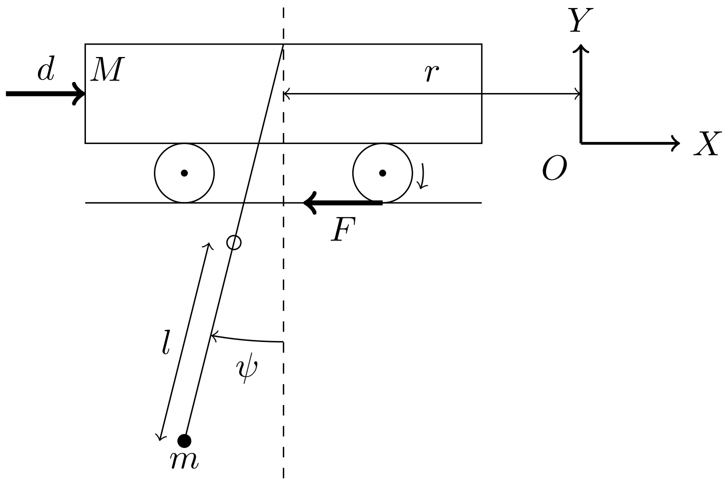

Figure 3: crane control example, note (O, X, Y) is a fixed frame

6.1 Model description

Notations are presented on Fig. 3. The system is controlled through its *DC* motor input u which produces a force F at the contact point (no-slip hypothesis). An unknown disturbance force d is eventually applied to the cart. The state of the system is given by $x = \begin{bmatrix} r & \psi & \dot{r} & \dot{\psi} & a_1 & a_2 \end{bmatrix}^\top$ where the last two variables describe the actuator dynamics. The model is linearised around $x = 0$. The regulated variable is defined as $\alpha = C_\alpha x = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} x = \psi$ where the objective is to satisfy $\forall t$, $|\alpha(t)| \leq 5°$ while driving the system from $r_0 = 0$ to the setpoint $r_\mathrm{s} = 10$m. Considering this regulated variable, it is observed that $l = 2$ and $k = 3 = l + 1$. The system state-space representation is given by:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1.089 & -0.1852 & 0 & 14.57 & 36.43 \\ 0 & -21.78 & 0.3704 & 0 & -29.14 & -72.86 \\ 0 & 0 & 0 & 0 & -12 & -5.005 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1.852 & 0 \\ -3.704 & 0 \\ 0 & 0.25 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

Using *LQR* design with $R = 10$ and $Q = \mathrm{diag}\left(\begin{bmatrix} 1 & 1 & 1 & 200 & 1 & 1 \end{bmatrix}\right)$, the following static state-feedback gain was obtained:

$$K = \begin{bmatrix} 0.3162 & 11.3982 & 0.5617 & -3.6933 & 23.4703 & 58.6667 \end{bmatrix}$$

This controller yields good results on r_s-setpoint tracking at the expense of huge variations on the angle ψ thus violating expected time-domain performance $|\psi| \leq 5°$. Introducing *OIST* in the loop, it is expected to comply with this constraint.

Note that a more precise selection of the controller K may deliver satisfying results. However, there may exist systems where opposed design constraints may be too hard to be satisfied together. As such, this example gives a complete methodology on how to implement *OIST* in such cases.

6.2 Hypotheses

- As already mentioned, considering the regulated variable $\alpha = \psi$, its relative degree wrt d equals $l = 2$ and $k = 3 = l + 1$ wrt to u. Hyp. 1 and 2 are thus satisfied;

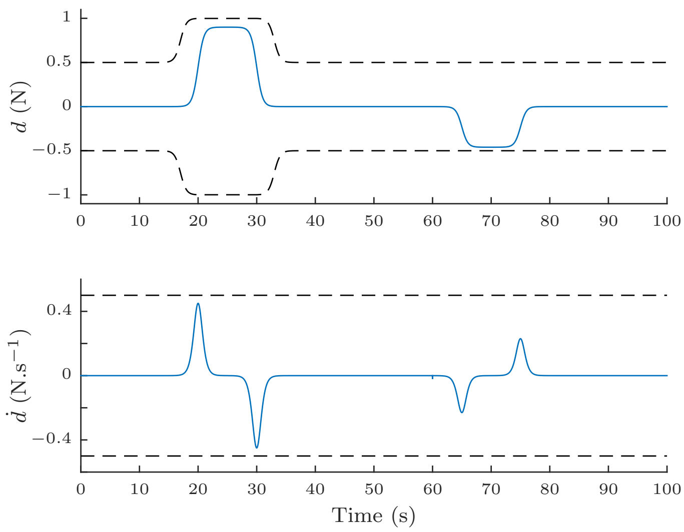

Figure 4: disturbance and disturbance derivative (in blue) along with known bounds (in black)

- As far as the unknown disturbance is concerned, the disturbance shown on Fig. 4 will be used for simulations with known bounds represented in red and green. Hyp. 3 is satisfied;
- Hyp. 4 and 5 are satisfied.

6.3 *OIST* implementation

Using results in Sect. 5 and considering $\alpha_{\min} = -\alpha_{\max} = -5°$, the expressions of Eq. (24) are obtained.

$$\begin{array}{rcl}
\alpha^{(3)} & = & C_\alpha A^3 x + C_\alpha A^2 B_u u + C_\alpha A^2 B_d d + C_\alpha A B_d \dot{d} \\
\alpha_{0,\min} & = & -\frac{5\pi}{180} \\
\alpha_{1,\min} & = & \kappa_1 \left(\alpha_{\min} - C_\alpha x\right) \\
\alpha_{2,\min} & = & \left(\kappa_2\kappa_1 + \dot{\kappa}_1\right)\left(\alpha_{\min} - C_\alpha x\right) - \left(\kappa_1 + \kappa_2\right) C_\alpha A x \\
\alpha_{3,\min} & = & \left(\kappa_3\kappa_2\kappa_1 + \kappa_3\dot{\kappa}_1 + \dot{\kappa}_1\kappa_2 + \dot{\kappa}_2\kappa_1 + \ddot{\kappa}_1\right)\left(\alpha_{\min} - C_\alpha x\right) - \left(\kappa_3\kappa_1 + \kappa_3\kappa_2 + \kappa_1\kappa_2\right. \\
 & & \left. +2\dot{\kappa}_1 + \dot{\kappa}_2\right) C_\alpha A x - \left(\kappa_1 + \kappa_2 + \kappa_3\right) C_\alpha A^2 x + \left(\kappa_1 + \kappa_2 + \kappa_3\right) |C_\alpha A B_d| \max\left(|\underline{d}|, |\overline{d}|\right)
\end{array} \tag{24}$$

As far as crossings are concerned and using the results from Eq. (24), the following holds:

$$\begin{array}{rcl}
\Omega_0 & = & \frac{\pi}{18} \\
\Omega_1 & = & \frac{\pi}{18}\kappa_1 \\
\Omega_2 & = & \frac{\pi}{18}\left(\kappa_2\kappa_1 + \dot{\kappa}_1\right) \\
\Omega_3 & = & \frac{\pi}{18}\kappa_3\kappa_2\kappa_1 - 2\left(\kappa_1 + \kappa_2 + \kappa_3\right)|C_\alpha A B_d| \max\left(|\underline{d}|, |\overline{d}|\right)
\end{array} \tag{25}$$

which results in the following coefficients where $(\breve{\kappa}_1, \breve{\kappa}_2, \breve{\kappa}_3)$ are strictly positive constants:

$$\begin{array}{rcl}
\kappa_1 & = & \frac{18}{\pi}\breve{\kappa}_1 \\
\kappa_2 & = & \frac{18}{\pi}\frac{\breve{\kappa}_2}{\breve{\kappa}_1} \text{ s.t.} \\
\breve{\kappa}_2 & > & \frac{\pi}{9}|C_\alpha A B_d| \max(|\underline{d}|, |\overline{d}|), \forall t \\
\kappa_3 & = & \frac{1}{\frac{18}{\pi}\breve{\kappa}_2 - 2|C_\alpha A B_d| \max(|\underline{d}|, |\overline{d}|)} \left(\breve{\kappa}_3 + \left[\frac{36}{\pi}\left(\breve{\kappa}_1 + \frac{\breve{\kappa}_2}{\breve{\kappa}_1}\right)|C_\alpha A B_d|\right.\right. \\
 & & \left.\left. +2|C_\alpha A^2 B_d|\right] \max(|\underline{d}|, |\overline{d}|) + 2|C_\alpha A B_d| \max(|\underline{\dot{d}}|, |\overline{\dot{d}}|)\right)
\end{array} \tag{26}$$

Note that coefficients κ_1 and κ_2 are constants due to the bounds $\alpha_{\min}$ and $\alpha_{\max}$ being constants. This would not be the case otherwise. Finally, the expressions for the saturations to apply to the controller output are detailed in Eq. (27).

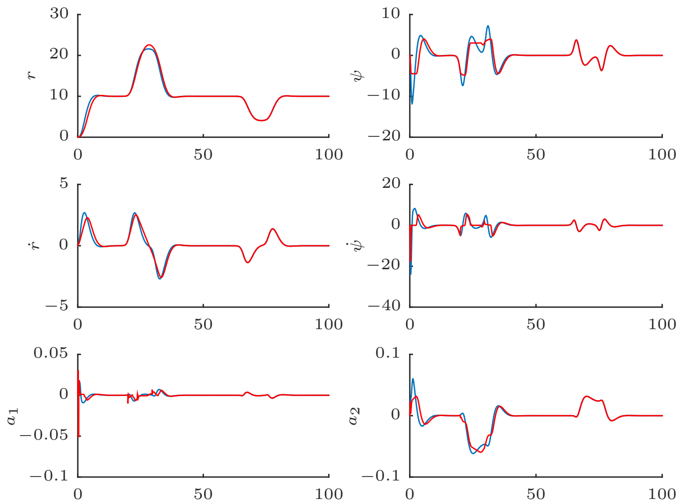

Figure 5: In red (resp. blue) state with (resp. without) *OIST* in the loop.

$$
\begin{aligned}
u_{\min} &= \tfrac{1}{C_\alpha A^2 B_u}\left[\kappa_3\kappa_2\kappa_1\left(\alpha_{\min} - C_\alpha x\right) - \left(\kappa_3\kappa_1 + \kappa_3\kappa_2 + \kappa_1\kappa_2\right) C_\alpha A x - \left(\kappa_1+\kappa_2+\kappa_3\right) C_\alpha A^2 x\right.\\
&\quad + \left(\kappa_1+\kappa_2+\kappa_3\right)|C_\alpha A B_d|\max(|\underline{d}|,|\overline{d}|) - C_\alpha A^3 x + |C_\alpha A B_d|\max(|\dot{\underline{d}}|,|\dot{\overline{d}}|)\\
&\quad \left. + |C_\alpha A^2 B_d|\max(|\underline{d}|,|\overline{d}|)\right]\\
u_{\max} &= \tfrac{1}{C_\alpha A^2 B_u}\left[\kappa_3\kappa_2\kappa_1\left(\alpha_{\max} - C_\alpha x\right) - \left(\kappa_3\kappa_1 + \kappa_3\kappa_2 + \kappa_1\kappa_2\right) C_\alpha A x - \left(\kappa_1+\kappa_2+\kappa_3\right) C_\alpha A^2 x\right.\\
&\quad - \left(\kappa_1+\kappa_2+\kappa_3\right)|C_\alpha A B_d|\max(|\underline{d}|,|\overline{d}|) - C_\alpha A^3 x - |C_\alpha A B_d|\max(|\dot{\underline{d}}|,|\dot{\overline{d}}|)\\
&\quad \left. - |C_\alpha A^2 B_d|\max(|\underline{d}|,|\overline{d}|)\right]
\end{aligned}
\tag{27}
$$

Note that $C_\alpha A^2 Bu < 0$ in this example. For the expressions in Eq. (27) to be still valid in simulation, one can obtain the control saturated value using:

$$
\operatorname{sat}(u) = \max\left(\min\left(u_{\min}, u_{\max}\right), \min\left(u, \max\left(u_{\min}, u_{\max}\right)\right)\right) \tag{28}
$$

6.4 Simulations and results

The simulations are performed over 100s using the following parameters: $\breve{\kappa}_1 = 1.5$, $\breve{\kappa}_2 = 150$ and $\breve{\kappa}_3 = 11000$. The disturbance force d shown on Fig. 4 is applied to the system. The second disturbance around $t = 70$s is used to confirm that when constraints are not violated then the command is not modified. Also note that using constant bounds on the unknown perturbation would yield good although more conservative results (and κ_3 would not depend on time).

Simulation results are shown on Fig. 5 to Fig. 7. As expected, the regulated variable satisfies the bounds constraint and the performance is slightly degraded on the position. Sharper variations are also witnessed on the command. The trade-off between performance and command energy can be tuned using the parameters $\breve{\kappa}$. Note that the regulated variable does not violate the constraint at time $t = 70$s despite the presence of a disturbance. Thus, the command is not modified and nominal performance is ensured.

7 Conclusions

The Output- to Input-Saturation Transformation (*OIST*) has been demonstrated and successfully applied to systems satisfying to Hyp. 1 to 5. Using an existing feedback gain and appropriate reformulation, the time-domain constraint $\forall t,\ \alpha_{\min}(t) \leq \alpha(t) \leq \alpha_{\max}(t)$ is enforced using saturations $\operatorname{sat}(u)$ on the controller output u: $u_{\min}(t) \leq u \leq u_{\max}(t)$.

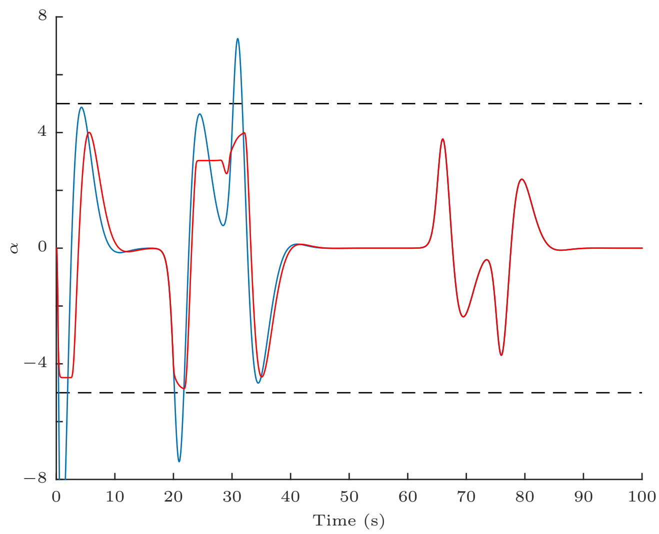

Figure 6: In red (resp. blue) regulated variable $\alpha = \psi$ with (resp. without) *OIST* in the loop. Expected bounds are represented in black.

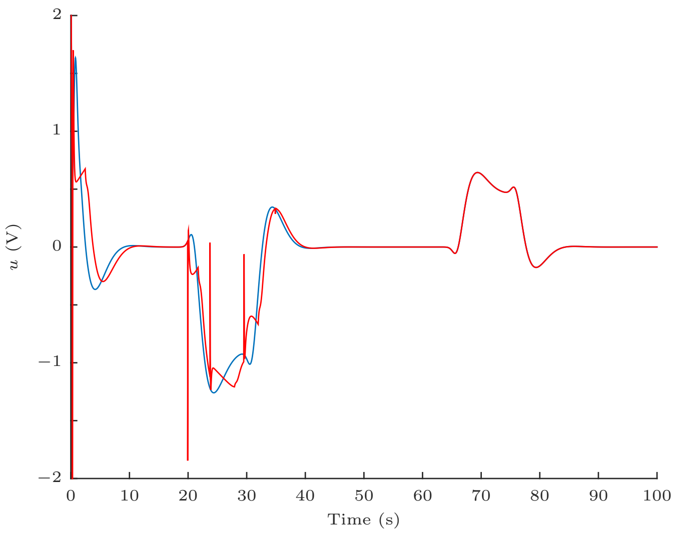

Figure 7: Saturated command in red. Maximal absolute value is 8.2V at $t = 0.22$s.

Note that stability proofs are being considered at the time of writing. Future works will address the output-feedback case, which has already been partially studied in [8]. The hypothesis $k = l + 1$ will also be weakened as well as the need for $T_{u \to \alpha}$ to only have strictly positive zeros.

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